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RESEARCH ARTICLE

Minimal but Inefficient Presentations of the Semi-Direct Products of Some Monoids

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Abstract

In recent papers of Ruskuc, Saito and J. Wang, the semi-direct product of two arbitrary monoids and a standard presentation, say $\mathcal{P}$, for this product have received considerable attention. Wang defined a trivialiser set of the Squier complex associated with $\mathcal{P}$ and after that necessary and sufficient conditions for $\mathcal{P}$ to be efficient have been given by Çevik. As a main result of this paper, we give sufficient conditions for a presentation of the semi-direct product of a one-relator monoid by an infinite cyclic monoid to be *minimal* but *not efficient*. In the final part of this paper we give some applications of this result.

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1. Introduction

1.1. Efficiency of monoids

A *monoid presentation*

$$\mathcal{P} = [\mathbf{y}; \mathbf{s}] \tag{1}$$

is a pair where $\mathbf{y}$ is a set (the generating symbols) and each $S \in \mathbf{s}$ (a relation) is an ordered pair (S_+, S_-), where S_+ and S_- are distinct, positive words on $\mathbf{x}$. We remark that one of S_+, S_- may be the empty positive word. One usually writes S: $S_+ = S_-$. Also it is said that $\mathcal{P}$ is finite if $\mathbf{y}$ and $\mathbf{s}$ are both finite. One can define a monoid $M(\mathcal{P})$ which is associated with $\mathcal{P}$ (in fact $M(\mathcal{P})$ is the quotient of $F(\mathbf{y})$ by the smallest congruence generated by $\mathbf{s}$ where $F(\mathbf{y})$ is the free monoid on $\mathbf{y}$). If W is a word on $\mathbf{y}$ then $\bar{W}$ denotes the element of $M(\mathcal{P})$. Throught this paper the notation M will be used instead of $M(\mathcal{P})$.

Let M be a monoid with a presentation $\mathcal{P}$ as in (1). Then the *Euler characteristic* of $\mathcal{P}$ is defined by $\chi(\mathcal{P}) = 1 - |\mathbf{y}| + |\mathbf{s}|$ where $|.|$ denotes the minimal number of elements in the set. Let

$$\delta(M) = 1 - rk_{\mathbb{Z}}(H_1(M)) + d(H_2(M)),$$

where $rk_{\mathbb{Z}}(.)$ denotes the $\mathbb{Z}$-rank of the torsion-free part and $d(.)$ means the minimal number of generators. Recently it has been shown by S. Pride (unpublished) there exists

$$\chi(\mathcal{P}) \geq \delta(M).$$

Then we define

$$\chi(M) = min\{\chi(\mathcal{P})\colon \mathcal{P} \text{ is a finite presentation for } M\}.$$

We should remark that some authors consider, just as with the group presentations, $-|\mathbf{y}| + |\mathbf{s}|$, and call this the *deficiency* of the presentation $\mathcal{P}$, denote by $def(\mathcal{P})$. The deficiency of a monoid M, denoted by $def(M)$, is then taken to be the minimum deficiency of any finite presentations of M. Clearly $1 + def(\mathcal{P}) = \chi(\mathcal{P})$ and $1 + def(M) = \chi(M)$.

Definition 1.1. Let M be a monoid.

i) A presentation $\mathcal{P}_0$ for M is called minimal if $\chi(\mathcal{P}_0) \leq \chi(\mathcal{P})$, for all presentations $\mathcal{P}$ of M.

ii) A finite presentation $\mathcal{P}$ is called efficient if $\chi(\mathcal{P}) = \delta(M)$.

iii) M is called efficient if $\chi(M) = \delta(M)$.

We note that not all monoids are efficient. Thus there is interest in finding *inefficient* finitely presented monoids. So we can give the following remark to define how to show that a monoid is inefficient.

Remark 1.2. If we can find a minimal presentation $\mathcal{P}$ for a monoid M such that $\mathcal{P}$ is not efficient then we have

$$\chi(\mathcal{P}') \geq \chi(\mathcal{P}) > \delta(M),$$

for all presentations $\mathcal{P}'$ defining the same monoid M. Thus there is no efficient presentation for M, that is, M is not an efficient monoid.

One can find some examples of efficient and inefficient monoid presentations, for instance, in [1], [2], [3] and [4].

1.2. The p-Cockcroft property of monoids

Let Γ be a graph associated with $\mathcal{P}$ (called Squier graph) which is defined as follows. The vertices are the elements of $F(\mathbf{y})$ and the edges are the 4-tuples $e = (U, S, \varepsilon, V)$ where $U, V \in F(\mathbf{y})$, $S \in \mathbf{s}$ and $\varepsilon = \pm 1$. The initial, terminal and inversion functions for an edge e as above are given by $\iota(e) = US_{\varepsilon}V$, $\tau(e) = US_{-\varepsilon}V$ and $e^{-1} = (U, S, -\varepsilon, V)$. There is an equivalence relation ([12]) on paths in Γ. Then Γ with this relation on paths, is called the Squier complex of $\mathcal{P}$ and denoted by $\mathcal{D}(\mathcal{P})$ (see, for example, in [12], [13], [16]). The elements of $\mathcal{D}(\mathcal{P})$ can be represented by geometric configurations, called *spherical monoid pictures*. These are described in detail in [12], [13] and we refer the reader these for details. Also, as described in [12], there are certain operations on spherical monoid pictures. Suppose $\mathbf{Y}$ is a collection of spherical monoid pictures over $\mathcal{P}$. Then, by [12], one can define an additional operation on spherical pictures.

Allowing this additional operation leads to the notion of equivalence (rel $\mathbf{Y}$) of spherical pictures. Then, by [13, Theorem 5.1], we say that $\mathbf{Y}$ *is a trivialiser of* $\mathcal{D}(\mathcal{P})$ *if and only if every spherical monoid picture is equivalent to an empty picture (rel* $\mathbf{Y}$*)*. Some examples and the details of the trivialiser can be found in [3], [6], [8], [12], [13], [16] and [18].

For any monoid picture $\mathbb{P}$ over $\mathcal{P}$ and for any $S \in \mathbf{s}$, $exp_S(\mathbb{P})$ denotes the *exponent sum* of S in $\mathbb{P}$ which is the number of positive discs labelled by S_+, minus the number of negative discs labelled by S_-. Moreover, for $y \in \mathbf{y}$, $L_y(S)$ denotes the *length* of S with respect to y.

Definition 1.3. Let $\mathcal{P}$ be as in (1) and let n be a non-negative integer. Then $\mathcal{P}$ is said to be n-Cockcroft if $exp_S(\mathbb{P}) \equiv 0 \pmod n$, (where congruence $(mod\, 0)$ is taken to be equality) for all $S \in \mathbf{s}$ and for all spherical pictures $\mathbb{P}$ over $\mathcal{P}$. A monoid M is said to be n-Cockcroft if it admits an n-Cockcroft presentation.

We remark that to verify the n-Cockcroft property holds, it is enough to check for pictures $\mathbb{P} \in \mathbf{Y}$, where $\mathbf{Y}$ is a set of generating pictures (see [12], [13]).

The 0-Cockcroft property is usually just called Cockcroft. In practice, we usually take n to be 0 or a prime p. The Cockcroft and p-Cockcroft properties of monoids have received considerable attention in [3]. Also the subjects *aspherical* and *combinatorial aspherical* have studied in [3], [5], [8, Section 5], [9], [13, Section 7] and [14] which imply the Cockcroft and then p-Cockcroft properties.

The following result has also recently been proved by S. Pride (unpublished).

Theorem 1.4. *Let* $\mathcal{P}$ *be as in* (1)*. Then* $\mathcal{P}$ *is efficient if and only if it is* p*-Cockcroft for some prime* p*.*

1.3. The definition and a presentation of semi-direct product of monoids

Let A, K be monoids and let θ be a monoid homomorphism

$$\theta\colon A \longrightarrow End(K), \quad a \longmapsto \theta_a \quad (a \in A), \quad 1 \longmapsto id_{End(K)}$$

where $End(K)$ denotes the collection of endomorphisms of K which is itself a monoid with identity $id\colon K \longrightarrow K$ (see [7] for some examples of the monoid endomorphisms). Then, by [15], [17] and [18], we can construct a monoid $B = K \rtimes_\theta A$ with elements (a,k) where $a \in A$ and $k \in K$ and product $(a,k)(a',k') = (aa', (k\theta_{a'})k')$.

Let $\mathcal{P}_A = [\mathbf{x};\mathbf{r}]$ and $\mathcal{P}_K = [\mathbf{y};\mathbf{s}]$ be presentations for A and K respectively. For each $y \in \mathbf{y}$ and $x \in \mathbf{x}$, let $y\theta_x$ denote a fixed positive word on $\mathbf{y}$ such that $[y\theta_x]_K = [y]_K\theta_{[x]_A}$ ($y\theta_x$ is unique modulo $\mathbf{s}$). Let T_{yx} denotes the relator

$yx = x(y\theta_x)$ and let $\mathbf{t}$ be the set of all relators of the form T_{yx} $(x \in \mathbf{x}, y \in \mathbf{y})$. Then, by [15], [17], [18],

$$\mathcal{P}_B = [\mathbf{x}, \mathbf{y}; \mathbf{r}, \mathbf{s}, \mathbf{t}] \tag{2}$$

is a presentation for B.

1.4. The p-Cockcroft property of the semi-direct product of arbitrary monoids

Throughout this section A, K will be denoted monoids with presentations $\mathcal{P}_A = [\mathbf{x}; \mathbf{r}]$ and $\mathcal{P}_K = [\mathbf{y}; \mathbf{s}]$, respectively. Also $\mathcal{P}_B$ will be denoted a presentation of the semi-direct product of K by A, as in (2).

In [18, Section 4], paths in Squier graph Γ have been used to construct a trivialiser set of $D(\mathcal{P}_B)$ where Γ is associated with $\mathcal{P}_B$. By [12], since every monoid picture over $\mathcal{P}_B$ can be represented by a path in Γ then the pictures will be used for a trivialiser set of $D(\mathcal{P}_B)$ (see [3]) for the rest of this paper.

Let $\mathbf{X_A}$ and $\mathbf{X_K}$ be trivialiser sets of $\mathcal{D}(\mathcal{P}_A)$ and $\mathcal{D}(\mathcal{P}_K)$, respectively.

If $W = y_1 y_2 \cdots y_m$ is a positive word on $\mathbf{y}$ then for any $x \in \mathbf{x}$, we denote the positive word $(y_1\theta_x)(y_2\theta_x)\cdots(y_m\theta_x)$ by $W\theta_x$. If $U = x_1 x_2 \cdots x_n$ is a positive word on $\mathbf{x}$ then for any $y \in \mathbf{y}$, we denote the positive word $(\cdots((y\theta_{x_1})\theta_{x_2})\theta_{x_3}\cdots)\theta_{x_n})$ by $y\theta_U$, and this can be represented by a monoid picture, say $\mathbb{A}_{U,y}$, as in Figure 1 (see also [3]).

Let $S \in \mathbf{s}, x \in \mathbf{x}$. Since $[S_+\theta_x]_{\mathcal{P}_K} = [S_-\theta_x]_{\mathcal{P}_K}$, there is a non-spherical picture, say $\mathbb{B}_{S,x}$, over $\mathcal{P}_K$ with $\iota(\mathbb{B}_{S,x}) = S_+\theta_x$ and $\tau(\mathbb{B}_{S,x}) = S_-\theta_x$. Note that, by the dependence on the choice of homomorphism θ_x, there are various $\mathbb{B}_{S,x}$ pictures which can be drawn.

Let $R \in \mathbf{r}, y \in \mathbf{y}$. Then we get non-spherical pictures $\mathbb{A}_{R_+,y}$ and $\mathbb{A}_{R_-,y}$, respectively, as in Figure 1. We should note that, these pictures consist of only T_{yx} discs $(x \in \mathbf{x})$. Moreover, since $[y\theta_{R_+}]_{\mathcal{P}_K} = [y\theta_{R_-}]_{\mathcal{P}_K}$, there is a non-spherical picture, say $\mathbb{C}_{y,\theta_R}$, over $\mathcal{P}_K$ with $\iota(\mathbb{C}_{y,\theta_R}) = y\theta_{R_+}$ and $\tau(\mathbb{C}_{y,\theta_R}) = y\theta_{R_-}$. We should also note that there are various $\mathbb{C}_{y,\theta_R}$ pictures which can be drawn for the same reason as above.

One can construct spherical monoid pictures, say $\mathbb{P}_{S,x}$ and $\mathbb{P}_{R,y}$ as shown in Figure 2, by using the non-spherical pictures $\mathbb{B}_{S,x}$, $\mathbb{A}_{R_+,y}$, $\mathbb{A}_{R_-,y}$ and $\mathbb{C}_{y,\theta_R}$ (see [3] for the details).

Let

$$\mathbf{C_1} = \{\mathbb{P}_{S,x} \colon S \in \mathbf{s}, x \in \mathbf{x}\} \text{ and } \mathbf{C_2} = \{\mathbb{P}_{R,y} \colon R \in \mathbf{r}, y \in \mathbf{y}\}.$$

The proof of the following lemma can be found in [18]. In fact this lemma is used in the proof of Theorem 1.6 below (see [4] for the details).

Lemma 1.5. *Suppose that $B = K \rtimes_\theta A$ is a semi-direct product with associated presentation $\mathcal{P}_B$, as in (2). Let $\mathbf{X_A}$ and $\mathbf{X_K}$ be trivialiser sets of the Squier complexes $\mathcal{D}(\mathcal{P}_A)$ and $\mathcal{D}(\mathcal{P}_K)$, respectively. Then a trivialiser set of $\mathcal{D}(\mathcal{P}_B)$ is*

$$\mathbf{X_A} \cup \mathbf{X_K} \cup \mathbf{C_1} \cup \mathbf{C_2}. \tag{3}$$

$\mathbb{A}_{U,y}$

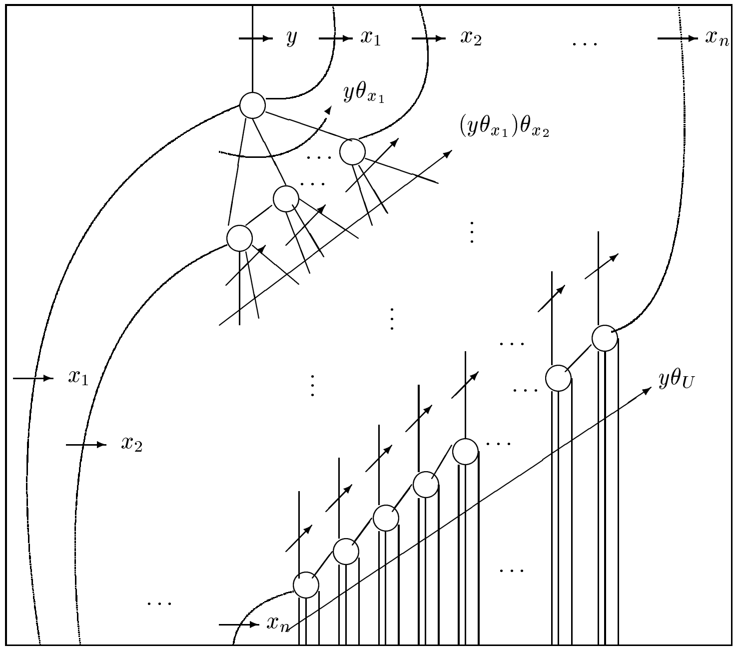


Figure 1:

Let us denote the set (3) by $\mathbf{X_B}$.

Theorem 1.6. *Let p be a prime or 0. Then the presentation $\mathcal{P}_B$, as in (2), is p-Cockcroft if and only if the following conditions hold.*

(i) *$\mathcal{P}_A$ and $\mathcal{P}_K$ are p-Cockcroft,*

(ii) $\exp_y(S) \equiv 0$ *(mod p) for all* $S \in \mathbf{s}, y \in \mathbf{y}$,

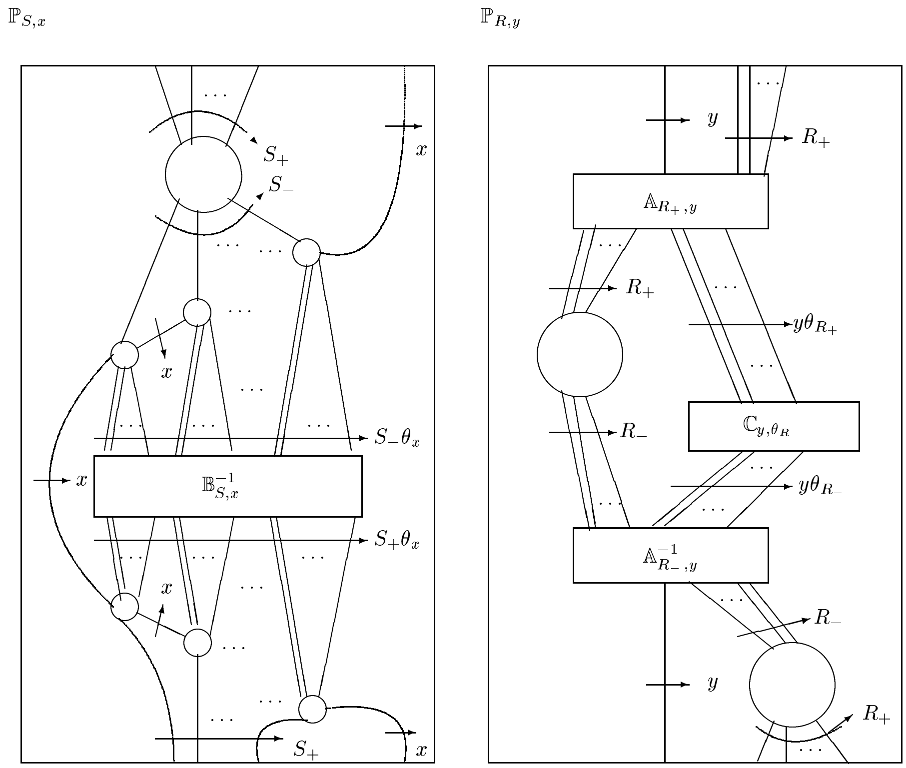
(iii) $\exp_{S_0}(\mathbb{B}_{S,x}) \equiv \begin{cases} 1, & S_0 = S \\ 0, & \textit{otherwise} \end{cases}$ *(mod p) for all* $S_0, S \in \mathbf{s}, x \in \mathbf{x}$,

(iv) $\exp_S(\mathbb{C}_{y,\theta_R}) \equiv 0$ *(mod p) for all* $S \in \mathbf{s}, y \in \mathbf{y}, R \in \mathbf{r}$,

(v) $\exp_{T_{yx}}(\mathbb{A}_{R_+,y}) \equiv \exp_{T_{yx}}(\mathbb{A}_{R_-,y})$ *(mod p) for all* $R \in \mathbf{r}, y \in \mathbf{y}$ *and* $x \in \mathbf{x}$.

2. The p-Cockcroft property of the semi-direct products of one-relator monoids by infinite cyclic monoids

Let K be a one-relator monoid with a presentation $\mathcal{P}_K = [\mathbf{y}; S_+ = S_-]$, and let A be the infinite cyclic monoid with a presentation $\mathcal{P}_A = [x;\ ]$. Let ψ be



(**a**) (**b**)

Figure 2:

an endomorphism of K, as in Section 1.3. Then, by [3], the mapping $x \longmapsto \psi$ induces a homomorphism θ: $A \longrightarrow End(K)$, and then we have a presentation

$$\mathcal{P}_B = [\mathbf{y}, x; S_+ = S_-, \mathbf{t}], \tag{4}$$

as in (2), for the monoid $B = K \rtimes_\theta A$ where $\mathbf{t}$ is the set of relators $T_{yx} (y \in \mathbf{y})$. Then, by Lemma 1.5, we have a trivializer set $\mathbf{X_B}$, as in (3). Notice that, since $\mathcal{P}_A$ is aspherical then, by [13], $\mathbf{X_A} = \emptyset$. Also, for the relator S, let us assume that $\iota(S_+) \neq \iota(S_-)$ (or $\tau(S_+) \neq \tau(S_-)$). So, by [8], $\mathcal{P}_K$ is aspherical, then $\mathbf{X_K} = \emptyset$. Moreover, since $\mathbf{r} = \emptyset$ then $\mathbf{C_2} = \emptyset$. Therefore $\mathbf{X_B} = \mathbf{C_1}$. Note that we have a single $\mathbb{P}_{S,x}$ picture, as in Figure 2-**(a)**, in the set $\mathbf{C_1}$ since K is a one-relator monoid.

As a consequence of Theorem 1.6, we have:

Corollary 2.1. *Let p be a prime or 0, and let K be a one-relator monoid, with relator S say. Suppose that $\iota(S_+) \neq \iota(S_-)$ (or $\tau(S_+) \neq \tau(S_-)$). Let B be a semi-direct product of K by an infinite cyclic monoid A with associated* *presentation $\mathcal{P}_B$, as in (4). Then $\mathcal{P}_B$ is p-Cockcroft if and only if*

(a) $\exp_y(S) \equiv 0$ *(mod p) for all* $y \in \mathbf{y}$,

(b) $\exp_S(\mathbb{B}_{S,x}) \equiv 1$ *(mod p)*.

Proof. Since $\mathcal{P}_A$ and $\mathcal{P}_K$ are aspherical and $\mathbf{C_2} = \emptyset$ then the conditions (i), (iv) and (v) of Theorem 1.6 are trivial. On the other hand, the condition (*ii*) gives (*a*) and the condition (iii) gives (b). Hence the result. ∎

3. The main theorem

As we mentioned in Theorem 1.4, a presentation is efficient if and only if it is p-Cockcroft, for some prime p. It follows from this result and Corollary 2.1 that the presentation $\mathcal{P}_B$, as in (4) is efficient if and only if there is a prime p such that

- $\exp_y(S) \equiv 0 (\text{mod } p)$ for all $y \in \mathbf{y}$,
- $\exp_S(\mathbb{B}_{S,x}) \equiv 1 \ (\text{mod } p)$,

in other words, if and only if

$$hcf(\exp_y(S)(y \in \mathbf{y}), \exp_S(\mathbb{B}_{S,x}) - 1) \neq 1.$$

In particular, $\mathcal{P}_B$ is *not efficient* if

$$\exp_S(\mathbb{B}_{S,x}) = 0 \text{ or } 2.$$

Let $d = hcf(\exp_y(S)(y \in \mathbf{y}))$. The value of d will be taken to be 0 if all exponent sums are 0 in $hcf(\exp_y(S)\colon y \in \mathbf{y})$.

Our **main result** of this paper is the following.

Theorem 3.1. *The presentation $\mathcal{P}_B$, as in (4), is* **minimal** *(but not efficient) if*

$$d \neq 2^n \quad and \quad \exp_S(\mathbb{B}_{S,x}) = 2,$$

for any $n \in \mathbb{Z}^+$.

4. The preliminaries and proof of the main theorem

Let M be a monoid with the presentation $\mathcal{P}$, as in (1). Let

$$P^{(l)} = \bigoplus_{S \in \mathbf{S}} \mathbb{Z}Me_S$$

be the free left $\mathbb{Z}M$-module with bases $\{e_S\colon S \in \mathbf{s}\}$. For an atomic monoid picture (see [12], [13] for the details), say $\mathbb{A} = (U, S, \varepsilon, V)$ where $U, V \in$

$F(\mathbf{y}), S \in \mathbf{s}, \varepsilon = \pm 1$, the left evaluation of the positive atomic monoid picture $\mathbb{A}$ is defined by

$$eval^{(l)}(\mathbb{A}) = \varepsilon \bar{U} e_S \in P^{(l)}$$

where $\bar{U} \in M$. For any spherical monoid picture $\mathbb{P} = \mathbb{A}_1 \mathbb{A}_2 \cdots \mathbb{A}_n$, where each $\mathbb{A}_i$ is an atomic picture for $i = 1, 2, \ldots, n$, we then define

$$eval^{(l)}(\mathbb{P}) = \sum_{i=1}^{n} eval^{(l)}(\mathbb{A}_i) \in P^{(l)}.$$

We let $\lambda_{\mathbb{P},S}$ be the coefficient of e_S in $eval^{(l)}(\mathbb{P})$, so we can write

$$eval^{(l)}(\mathbb{P}) = \sum_{S \in \mathbf{S}} \lambda_{\mathbb{P},S} e_S \in P^{(l)}.$$

Let $I_2^{(l)}(\mathcal{P})$ be the 2-sided ideal of $\mathbb{Z}M$ generated by the set

$$\{\lambda_{\mathbb{P},S} \colon \mathbb{P} \text{ is a spherical monoid picture}, S \in \mathbf{s}\}.$$

Then this ideal is called the *second Fox ideal* of $\mathcal{P}$. The concept of Fox ideals for the monoids has been discussed in [3].

Remark 4.1. If $\mathbf{Y}$ is a trivializer of $\mathcal{D}(\mathcal{P})$ then $I_2^{(l)}(\mathcal{P})$ is generated (as a 2-sided ideal) by the set

$$\{\lambda_{\mathbb{P},S} \colon \mathbb{P} \in \mathbf{Y}, S \in \mathbf{s}\}.$$

We remark that one can also give the definition of the second Fox ideals by using the free right $\mathbb{Z}M$-module.

In fact we need the concept of the second Fox ideals for Theorem 4.2 below which is recently proved by S. Pride (unpublished) and is a test of minimality of monoid presentations. We remark that the group presentation version of this theorem can also be found in [10] and [11].

Let ψ be a ring homomorphism from $\mathbb{Z}M$ into the ring of all $k \times k$ matrices over a commutative ring L with 1, for some $k \geq 1$, and suppose $\psi(1) = I_{k \times k}$.

Theorem 4.2. *Let $\mathbf{Y}$ be a trivializer of $\mathcal{D}(\mathcal{P})$. If $\psi(\lambda_{\mathbb{P},S}) = 0$ for all $\mathbb{P} \in \mathbf{Y}, S \in \mathbf{s}$ then $\mathcal{P}$ is minimal.*

It is clear that the above theorem can be restated by *if there is a ring homomorphism ψ as above such that $I_2^{(l)}(\mathcal{P})$ is contained in the kernel of ψ, then $\mathcal{P}$ is minimal.*

At the rest of the paper K will be denote a one-relator monoid with a presentation $\mathcal{P}_K = [\mathbf{y}; S_+ = S_-]$ and A will be denote the infinite cyclic monoid with a presentation $\mathcal{P}_A = [x;\ ]$. Also B will be denote the semi-direct product of K by A with a presentation $\mathcal{P}_B$, as given in (4).

Let us consider the picture $\mathbb{P}_{S,x}$, as in Figure 2-(**a**).

For a fixed $y \in \mathbf{y}$, let us assume that $\frac{\partial}{\partial y}$ denotes the Fox derivation with respect to y, and let $\frac{\partial^B}{\partial y}$ be the composition

$$\mathbb{Z}F(\mathbf{y}) \xrightarrow{\frac{\partial}{\partial y}} \mathbb{Z}F(\mathbf{y}) \longrightarrow \mathbb{Z}B,$$

where $F(\mathbf{y})$ is the free monoid on $\mathbf{y}$. Moreover, for the relator $S \in \mathbf{s}$, let us define $\frac{\partial^B S}{\partial y}$ to be

$$\frac{\partial^B S_+}{\partial y} - \frac{\partial^B S_-}{\partial y}.$$

Again for a fixed $y \in \mathbf{y}$, let us write

$$S_+ = U_0 y U_1 y \cdots U_{r-1} y U_r \text{ and } S_- = V_0 y V_1 y \cdots V_{k-1} y V_k,$$

where each U_i $(1 \leq i \leq r)$ and $V_j (1 \leq j \leq k)$ is a word on $\mathbf{y} - \{y\}$. Then, for this particular y, the left evaluations of the positive atomic pictures in $\mathbb{P}_{S,x}$ containing a T_{yx} disc are

$$\overline{U_0} e_{T_{yx}}, \overline{U_0 y U_1} e_{T_{yx}}, \ldots, \overline{U_0 y \cdots U_{r-1}} e_{T_{yx}},$$

and the left evaluations of the negative atomic pictures in $\mathbb{P}_{S,x}$ containing a T_{yx} disc are

$$-\overline{V_0} e_{T_{yx}}, -\overline{V_0 y V_1} e_{T_{yx}}, \ldots, -\overline{V_0 y \cdots V_{r-1}} e_{T_{yx}}.$$

Hence, for a fixed y, the coefficient of $e_{T_{yx}}$ in $eval^{(l)}(\mathbb{P}_{S,x})$ is

$$\overline{U_0} + \overline{U_0 y U_1} + \cdots + \overline{U_0 y \cdots U_{r-1}} - (\overline{V_0} + \overline{V_0 y V_1} + \cdots + \overline{V_0 y \cdots V_{r-1}}) = \frac{\partial^B S}{\partial y}. \quad (5)$$

Lemma 4.3. *The second Fox ideal $I_2^{(l)}(\mathcal{P}_B)$ of $\mathcal{P}_B$ is generated by the elements*

$$1 - \bar{x}(eval^{(l)}(\mathbb{B}_{S,x})) \text{ and } \frac{\partial^B S}{\partial y}(y \in \mathbf{y}).$$

Proof. Since $\mathcal{D}(\mathcal{P}_B)$ has a trivialiser $\mathbf{X_B}$ consisting of the single picture $\mathbb{P}_{S,x}$, we need to consider $eval^{(l)}(\mathbb{P}_{S,x})$. We have

$$eval^{(l)}(\mathbb{P}_{S,x}) = \lambda_{\mathbb{P}_{S,x},S} e_S + \sum_{y \in \mathbf{y}} \lambda_{\mathbb{P}_{S,x},T_{yx}} e_{T_{yx}},$$

where

$$\lambda_{\mathbb{P}_{S,x},S} = (1 - \bar{x}(eval^{(l)}(\mathbb{B}_{S,x}))) \text{ and } \lambda_{\mathbb{P}_{S,x},T_{yx}} = \frac{\partial^B S}{\partial y}(y \in \mathbf{y}) \text{ by (5).}$$

Thus, by Remark 4.1, we get the result. ■

Let

$$aug\colon\ \mathbb{Z}B \longrightarrow \mathbb{Z}, \quad b \longmapsto 1$$

be the augmentation map. By the meaning of this, we have the following lemma.

Lemma 4.4.

$$aug(eval^{(l)}(\mathbb{B}_{S,x})) = \exp_S(\mathbb{B}_{S,x}).$$

Proof. We can write

$$eval^{(l)}(\mathbb{B}_{S,x}) = \varepsilon_1 \overline{W_1} e_S + \varepsilon_2 \overline{W_2} e_S + \cdots + \varepsilon_n \overline{W_n} e_S,$$

where $\varepsilon_i = \pm 1$ and the W_i's are certain words on $\mathbf{y}$ $(1 \leq i \leq n)$. In the above expression, each term $\varepsilon_i \overline{W_i} e_S$ corresponds to a single S-disc. Also, the value of each ε_i gives the sign of this single S-disc. Therefore the sum of the ε_i's, that is, $aug(eval^{(l)}(\mathbb{B}_{S,x}))$ must give the exponent sum of the S-discs in the picture $\mathbb{B}_{S,x}$, as required. ■

The proof of the following lemma can be found in [3].

Lemma 4.5.

$$aug\left(\frac{\partial^B S}{\partial y}\right) = \exp_y(S) \quad (y \in \mathbf{y}).$$

Now we can prove our **main theorem** as follows.

Suppose that d is not equal to 2^n $(n \in \mathbb{Z}^+)$. Let

$$\mathbb{Z}_d = \begin{cases} \mathbb{Z} & d = 0 \\ \mathbb{Z} \pmod d & d \neq 0 \end{cases}.$$

Suppose also that $\exp_S(\mathbb{B}_{S,x}) = 2$.

Let us consider the homomorphism from B onto the infinite cyclic monoid generated by x, defined by

$$y \longmapsto 1 (y \in \mathbf{y}), x \longmapsto x.$$

This induces a ring homomorphism

$$\gamma\colon\ \mathbb{Z}B \longrightarrow \mathbb{Z}[x].$$

Note that the restriction of γ to the subring $\mathbb{Z}K$ of $\mathbb{Z}B$ is just the augmentation map

$$aug\colon\ \mathbb{Z}K \longrightarrow \mathbb{Z}.$$

Thus, by Lemmas 4.4 and 4.5, the image of $I_2^{(l)}(\mathcal{P}_B)$ under γ is the ideal of $\mathbb{Z}[x]$ generated by

$$1 - \bar{x}(\exp_S(\mathbb{B}_{S,x})) = 1 - 2\bar{x}, \exp_y(S)(y \in \mathbf{y}).$$

Let η be the composition of γ and the mapping

$$\mathbb{Z}[x] \longrightarrow \mathbb{Z}_d[x], \quad x \longmapsto x, n \longmapsto \bar{n}(n \in \mathbb{Z}),$$

where $\bar{n}$ is $n \pmod d$. Then, since $\exp_y(S) \equiv 0 \pmod d (y \in \mathbf{y})$, we get

$$\begin{aligned} \eta(I_2^{(l)}(\mathcal{P}_B)) &= \langle 1 - \bar{2}\bar{x} \rangle \\ &= I, \text{ say.} \end{aligned}$$

Lemma 4.6.

$$I \neq \mathbb{Z}_d[x].$$

Proof. For simplicity, we shall replace $\bar{x}$ by x and $\bar{2}$ by 2. Thus we have $I = \langle 1 - 2x \rangle$. Then

$$\langle 1 - 2x \rangle = \{p(x)(1 - 2x)\colon \ p(x) \in \mathbb{Z}_d[x]\}. \tag{6}$$

Suppose that $\langle 1-2x \rangle = \mathbb{Z}_d[x]$ or equivalently, $1 \in I$. So, $1 = (1-2x)p(x)$ for some polynomial $p(x) \in \mathbb{Z}_d[x]$. Write $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_rx^r$ where $a_0, a_1, a_2, \ldots, a_r \in \mathbb{Z}_d$. Then

$$1 = a_0 + (a_1 - 2a_0)x + (a_2 - 2a_1)x^2 + \cdots + (a_r - 2a_{r-1})x^r - 2a_rx^{r+1}.$$

Thus $a_0 - 1 \equiv 0 \pmod d, a_1 - 2a_0 \equiv 0 \pmod d, \ldots, a_r - 2a_{r-1} \equiv 0 \pmod d$ and $-2a_r \equiv 0 \pmod d$. Since $d \neq 1, 2^n$, we can choose an *odd* prime p such that $p \mid d$. So, $p \mid -a_r$ (since p is odd then p does not divide 2, but we know that $p \mid d$ and $-2a_r \equiv 0 \pmod d$ then $p \mid -a_r$). Also, since $p \mid d, a_r - 2a_{r-1} \equiv 0 \pmod d$ then $p \mid -2a_{r-1} \Rightarrow p \mid -a_{r-1}$. Similarly, since $p \mid d$ and $a_{r-1} - 2a_{r-2} \equiv 0 \pmod d$ then $p \mid -2a_{r-2} \Rightarrow p \mid -a_{r-2}$. By iterating this procedure, we get $p \mid a_0$. Thus, since $p \mid d$ and $a_0 - 1 \equiv 0 \pmod d$ then $p \mid 1$. But it is a contradiction. Therefore $\langle 1 - 2x \rangle \neq \mathbb{Z}_d[x]$, as required. ■

Let ψ be the composition

$$\mathbb{Z}B \xrightarrow{\eta} \mathbb{Z}_d[x] \xrightarrow{\phi} \mathbb{Z}_d[x]/I,$$

where ϕ is the natural epimorphism. Then ψ sends $I_2^{(l)}(\mathcal{P}_B)$ to 0, and $\psi(1) = 1$. In other words, the images of the generators of $I_2(\mathcal{P}_B)$ are all 0 under ψ. That is,

$$\begin{aligned} \psi(1 - \bar{x}(eval^{(l)}(\mathbb{B}_{S,x}))) &= \phi\eta(1 - \bar{x}(eval^{(l)}(\mathbb{B}_{S,x}))) \\ &= \phi(1 - \bar{x}(\overline{\exp_S(\mathbb{B}_{S,x})})) \text{ since } \eta \text{ is a ring} \\ &\quad\ \text{homomorphism and by Lemma 4.4} \\ &= \phi(1 - \bar{x}\bar{2}) \text{ since } \exp_S(\mathbb{B}_{S,x}) = 2 \\ &= 0, \end{aligned}$$

and, for all $y \in \mathbf{y}$

$$\begin{aligned} \psi\left(\frac{\partial^B S}{\partial y}\right) &= \phi\eta\left(\frac{\partial^B S}{\partial y}\right) \\ &= \phi(\overline{\exp_y(S)}) \text{ since } \eta \text{ is a ring} \\ &\quad \text{homomorphism and by Lemma 4.5} \\ &= \phi(0) \text{ since } \exp_y(S) \equiv 0 \pmod d \\ &= 0. \end{aligned}$$

So, by Theorem 4.2 (Pride), $\mathcal{P}_B$ is minimal (and so, by Remark 1.2, B is a minimal monoid). Hence the result. ∎

Again for simplicity, let us replace $\bar{x}$ by x and $\bar{2}$ by 2.

Remark 4.7. Suppose that $d = 2^n (n \in \mathbb{Z}^+)$. Then we get $1 \in \langle 1 - 2x \rangle$, and so $\langle 1 - 2x \rangle = \mathbb{Z}_d[x]$.

(To see this it is enough to show $2 \in I = \langle 1 - 2x \rangle$, because we certainly have $1 - 2x \in I$ and if $2 \in I$ then we must have $1 \in I$. So let us take $1 - 2x \in I$. Then, by (6), we have

$$\begin{aligned} &2^{n-1}(1-2x) \in I \Rightarrow 2^{n-1} - 2^n x \in I = 2^{n-1} \in I \text{ since } 2^n x = 0 \text{ in } \mathbb{Z}_d[x] \Rightarrow \\ &2^{n-2}(1-2x) \in I \Rightarrow 2^{n-2} - 2^{n-1}x \in I \Rightarrow 2^{n-2} \in I \\ &\quad \text{since } 2^{n-1} \in I \text{ by the above line} \Rightarrow \\ &\cdots \text{ by iterating this procedure, we get } \cdots \Rightarrow 2 \in I \Rightarrow 1 \in I, \end{aligned}$$

as required.)

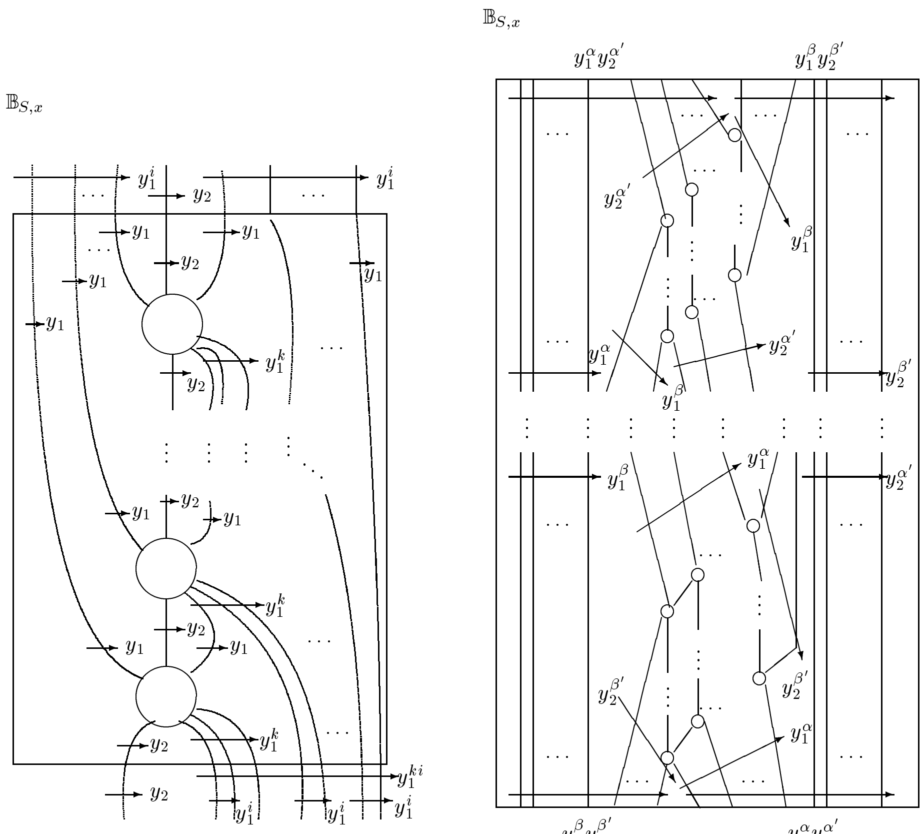
5. Some applications

In this section we will consider some examples of Theorem 3.1.

Let K be the free abelian monoid of rank 2, presented by $\mathcal{P}_K = [y_1, y_2; y_1 y_2 = y_2 y_1]$, and let ψ be the endomorphism $\psi_{\mathbf{M}}$ of K where $\mathbf{M}$ is the matrix $\begin{bmatrix} \alpha & \alpha' \\ \beta & \beta' \end{bmatrix}$ $(\alpha, \alpha', \beta, \beta' \in \mathbb{Z}^+)$, given by $[y_1] \longmapsto [y_1^{\alpha} y_2^{\alpha'}]$ and $[y_2] \longmapsto [y_1^{\beta} y_2^{\beta'}]$ (see [3, Examples 4.3.5 and 4.3.5.(*a*)]). Thus we have the presentation

$$\mathcal{P}_B = [y_1, y_2, x; S, T_{y_1 x}, T_{y_2 x}], \tag{7}$$

as in (4), for the monoid $B = K \rtimes_\theta A$ where S: $y_1 y_2 = y_2 y_1$, $T_{y_1 x}$: $y_1 x = x y_1^{\alpha} y_2^{\alpha'}$



(**a**) (**b**)

Figure 3:

and $T_{y_2 x}$: $y_2 x = x y_1^{\beta} y_2^{\beta'}$, respectively. Note that, by [3], the picture $\mathbb{B}_{S,x}$ can be given by Figure 3-**(b)**. In this $\mathbb{B}_{S,x}$ picture, since we have $\alpha\beta'$-times positive and $\alpha'\beta$-times negative S-discs then

$$\exp_S(\mathbb{B}_{S,x}) = \alpha\beta' - \alpha'\beta = \det \mathbf{M}.$$

Thus, as a consequence of Corollary 2.1, it is easy to see that the presentation $\mathcal{P}_B$, as in (7) is p-Cockcroft (for any prime p or 0) if and only if $\det \mathbf{M} \equiv 1 \pmod p$. Moreover since $\exp_{y_1}(S) = \exp_{y_2}(S) = 0$ then we get $d = 0$.

Thus, as a consequence of Theorem 3.1, we get

Corollary 5.1. *The presentation $\mathcal{P}_B$, as in (7) is minimal but not efficient if* $\det \mathbf{M} = 2$.

Example 5.2. One can choose the matrix $\mathbf{M} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$. Then we get a presentation $\mathcal{P}_B$, as in (7), for the monoid $B = K \rtimes_\theta A$ where

$$S\colon\ y_1y_2 = y_2y_1, T_{y_1x}\colon\ y_1x = xy_1^3y_2 \text{ and } T_{y_2x}\colon\ y_2x = xy_1y_2,$$

respectively. Thus, by Corollary 5.1, $\mathcal{P}_B$ is minimal.

Let K be the one-relator monoid with the presentation $\mathcal{P}_K = [y_1, y_2; S]$, where $S\colon\ y_1y_2y_1 = y_2y_1^k$, and let ψ_x be the endomorphism given by $[y_1] \longmapsto [y_1^i]$ and $[y_2] \longmapsto [y_2]$, where $i \in \mathbb{Z}^+$ (see [3, Example 4.2.16.(a)]). We then get the presentation

$$\mathcal{P}_B = [y_1, y_2, x; S, y_1x = xy_1^i, y_2x = xy_2], \tag{8}$$

as in (4), for the monoid $B = K \rtimes_\theta A$. By [3], the picture $\mathbb{B}_{S,x}$ can be given by Figure 3- **(a)**. Notice that in the picture $\mathbb{B}_{S,x}$, $exp_S(\mathbb{B}_{S,x}) = i$. Then, as a consequence of Corollary 2.1, it has been proved in [4] that *the presentation* $\mathcal{P}_B$*, as in (8) is* p*-Cockcroft (for any prime* p *or* 0*) if and only if* $k \equiv 2\ (mod\ p)$ *and* $i \equiv 1 (mod\ p)$ and so if $k = 2$ and $i = 1$ then $\mathcal{P}_B$ is 0-Cockcroft. But the condition $i = 1$ implies that ψ_x is the identity map and so θ is the trivial homomorphism. Then the presentation $\mathcal{P}_B$ becomes a presentation

$$[\mathbf{y}, x; \mathbf{s}, yx = xy (y \in \mathbf{y})]$$

which presents the direct product $K \times \mathbb{Z}^+$. Thus, by [4, Example 4.3], one can say immediately $\mathcal{P}_B$, as in (8) is 0-Cockcroft when $k = 2$ and $i = 1$.

Notice also that here we have $\exp_{y_1}(S) = 2 - k, \exp_{y_2}(S) = 0$. So $d = k - 2$. Then, as a consequence of Theorem 3.1, we have the following result.

Corollary 5.3. *The presentation* $\mathcal{P}_B$*, as in (8) is minimal (but inefficient) if*

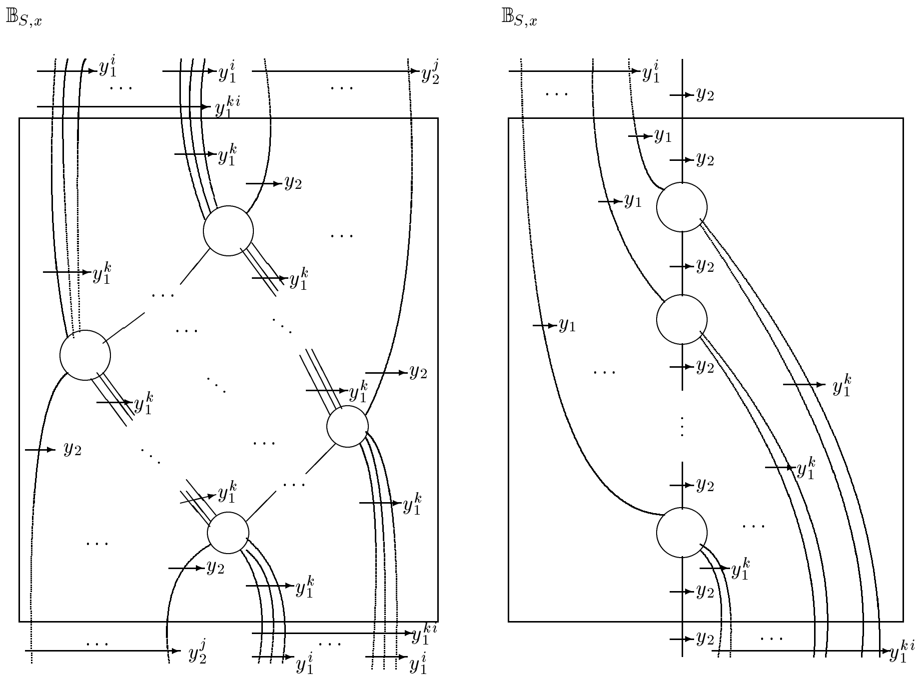
$$k \neq 2(2^{n-1} + 1) \quad \text{and} \quad i = 2,$$

where $n \in \mathbb{Z}^+$.

Let K be given by the presentation $\mathcal{P}_K = [y_1, y_2; S]$, where $S\colon\ y_1^ky_2 = y_2y_1^k$, and let ψ_x be the endomorphism given by $[y_1] \longmapsto [y_1^i]$ and $[y_2] \longmapsto [y_2^j]$, where $i, j \in \mathbb{Z}^+$ (see [3, Example 4.2.16.(b)] for the details). We then have a presentation

$$\mathcal{P}_B = [y_1, y_2, x; S, y_1x = xy_1^i, y_2x = xy_2^j], \tag{9}$$

as in (4), for the monoid $B = K \rtimes_\theta A$. By [3], the picture $\mathbb{B}_{S,x}$ can be given by Figure 4-**(a)**. Notice that $\exp_S(\mathbb{B}_{S,x}) = ij$ for the picture $\mathbb{B}_{S,x}$. Thus, as a consequence of Corollary 2.1, it has been given in [4] that *the presentation* $\mathcal{P}_B$*, as in (9) is* p*-Cockcroft (for any prime* p *or* 0*) if and only if* $ij \equiv 1 (mod\ p)$ and, as an example of this, if $i = j = 1$ then $\mathcal{P}_B$ is 0-Cockcroft. But as with



(a) (b)

Figure 4:

the previous example, the presentation $\mathcal{P}_B$ becomes a presentation of the direct product $K \times \mathbb{Z}^+$ when $i = j = 1$ holds. Then, by [4], one can say straightaway that the presentation $\mathcal{P}_B$ is 0-Cockcroft.

Notice that we have $\exp_{y_1}(S) = \exp_{y_2}(S) = 0$. So $d = 0$. Then, as an application of Theorem 3.1, the minimality of $\mathcal{P}_B$ can be given as follows.

Corollary 5.4. *The presentation $\mathcal{P}_B$, as in (9) is minimal (but inefficient) if*

$$(i, j) = (1, 2), (2, 1).$$

A similar example can be given as follows.

Let K be given by the presentation $\mathcal{P}_K = [y_1, y_2; S]$, where S: $y_1 y_2 = y_2 y_1^k$, and let ψ_x be the endomorphism given by $[y_1] \longmapsto [y_1^i]$ and $[y_2] \longmapsto [y_2]$, where $i \in \mathbb{Z}^+$ (see [3, Example 4.2.16.(c)]). Thus we have a presentation

$$\mathcal{P}_B = [y_1, y_2, x; S, y_1 x = x y_1^i, y_2 x = x y_2], \tag{10}$$

as in (4), for the monoid $B = K \rtimes_\theta A$. Again by [3], the picture $\mathbb{B}_{S,x}$ can be given by Figure 4-**(b)**. We note that $exp_S(\mathbb{B}_{S,x}) = i$. Then, it has been proved in [4] that *the presentation $\mathcal{P}_B$, as in (10) is p-Cockcroft (for any prime p and*

0) *if and only if* $k \equiv 1 \ (mod\, p)$ *and* $i \equiv 1 \ (mod\, p)$. Also, by [4], if $i = k = 1$ then $\mathcal{P}_B$ is 0-Cockcroft. But as previously, the presentation $\mathcal{P}_B$ becomes a presentation of the direct product $K \times \mathbb{Z}^+$ when $i = k = 1$ holds. Therefore one can say previously the presentation $\mathcal{P}_B$ is 0-Cockcroft.

Clearly we have $\exp_{y_1}(S) = 1 - k, \exp_{y_2}(S) = 0$. So that $d = k - 1$. Thus, again as an application of Theorem 3.1, we get the following result.

Corollary 5.5. *The presentation* $\mathcal{P}_B$*, as in (10) is minimal (but inefficient) if*

$$k \neq 2^n + 1 \text{ and } i = 2,$$

where $n \in \mathbb{Z}^+$.

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