Minimal but inefficient presentations for semi-direct products of finite cyclic monoids

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Abstract

Let A and K be arbitrary two monoids. For any connecting monoid homomorphism $\theta : A \longrightarrow End(K)$, let $M = K \rtimes_{\theta} A$ be the corresponding monoid semi-direct product. In [3], Cevik discussed necessary and sufficient conditions for the standard presentation of M to be efficient (or, equivalently, p-Cockcroft for any prime p or 0), and then, as an application of this, he showed the efficiency for the presentation, say $\mathcal{P}_M$, of the semi-direct product of any two finite cyclic monoids. As a main result of this paper, we give sufficient conditions for $\mathcal{P}_M$ to be *minimal* but not *efficient*. To do that we will use the same method as given in [4].

2000 *Mathematics Subject Classification*: 20L05, 20M05, 20M15, 20M50, 20M99.
Keywords and phrases: Minimality, Efficiency, p-Cockcroft property, Finite cyclic monoids.

1 Introduction

Let $\mathcal{P} = [X\ ;\ \mathbf{r}]$ be a monoid presentation where a typical element $R \in \mathbf{r}$ has the form $R_+ = R_-$. Here R_+, R_- are words on X (that is, elements of the free monoid $F(X)$ on X). The *monoid defined by* $[X\ ;\ \mathbf{r}]$ is the quotient of $F(X)$ by the smallest congruence generated by $\mathbf{r}$.

We have a (Squier) graph $\Gamma = \Gamma(X; \mathbf{r})$ associated with $[X\ ;\ \mathbf{r}]$, where the vertices are the elements of $F(X)$ and the edges are the 4-tuples $e = (U, R, \varepsilon, V)$ where $U, V \in F(X)$, $R \in \mathbf{r}$ and $\varepsilon = \pm 1$. The initial, terminal and inversion functions for an edge e as given above are defined by $\iota(e) = UR_{\varepsilon}V$, $\tau(e) = UR_{-\varepsilon}V$ and $e^{-1} = (U, R, -\varepsilon, V)$.

Two paths π and π' in a 2-complex are equivalent if there is a finite sequence of paths $\pi = \pi_0, \pi_1, \cdots, \pi_m = \pi'$ where for $1 \leq i \leq m$ the path π_i is obtained from π_{i-1} either by inserting or deleting a pair ee^{-1} of inverse edges or else by inserting or deleting a defining path for one of the 2-cells of the complex. There is an equivalence relation, $\sim$, on paths in Γ which is generated by $(e_1.\iota(e_2))(\tau(e_1).e_2) \sim (\iota(e_1).e_2)(e_1.\tau(e_2))$ for any edges e_1 and e_2 of Γ. This corresponds to requiring that the closed paths $(e_1.\iota(e_2))(\tau(e_1).e_2)(e_1^{-1}.\tau(e_2))(\iota(e_1).e_2^{-1})$ at the vertex $\iota(e_1)\iota(e_2)$ are the defining paths for the 2-cells of a 2-complex having Γ as its 1-skeleton. This 2-complex is called the Squier complex of $\mathcal{P}$ and denoted by $\mathcal{D}(\mathcal{P})$ (see, for example, [7], [12], [13], [15]). The paths in $\mathcal{D}(\mathcal{P})$ can be represented by geometric configurations, called *monoid pictures*. Monoid pictures and group pictures have been used in several papers by S. Pride and other authors. We assume here that the reader is familiar with monoid pictures (see [7, Section 4], [12, Section 1] or [13, Section 2]). Typically, we will use blackboard bold, e.g. $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{P}$, as notation for monoid pictures. Atomic monoid pictures are pictures which correspond to paths of length 1. Write $[|U, R, \varepsilon, V|]$ for the atomic picture which corresponds to the edge (U, R, ε, V) of the Squier complex. Whenever we can concatenate two paths π and π' in Γ to form the path $\pi\pi'$, then we can concatenate the corresponding monoid pictures $\mathbb{P}$ and $\mathbb{P}'$ to form a monoid picture $\mathbb{P}\mathbb{P}'$ corresponding to $\pi\pi'$. The equivalence of paths in the Squier complex corresponds to an equivalence of monoid pictures. That is, two monoid pictures $\mathbb{P}$ and $\mathbb{P}'$ are equivalent if there is a finite sequence of monoid pictures $\mathbb{P} = \mathbb{P}_0, \mathbb{P}_1, \cdots, \mathbb{P}_m = \mathbb{P}'$ where, for $1 \leq i \leq m$, the monoid picture $\mathbb{P}_i$ is obtained from the picture $\mathbb{P}_{i-1}$ either by inserting or deleting a subpicture $\mathbb{A}\mathbb{A}^{-1}$ where $\mathbb{A}$ is an atomic monoid picture or else by replacing a subpicture $(\mathbb{A}.\iota(\mathbb{B}))(\tau(\mathbb{A}).\mathbb{B})$ by $(\iota(\mathbb{A}).\mathbb{B})(\mathbb{A}.\tau(\mathbb{B}))$ or vice versa, where $\mathbb{A}$ and $\mathbb{B}$ are atomic monoid pictures.

A monoid picture is called a *spherical* monoid picture when the corresponding path in the Squier complex is a closed path. Suppose $\mathbf{Y}$ is a collection of spherical monoid pictures over $\mathcal{P}$. Two monoid pictures $\mathbb{P}$ and $\mathbb{P}'$ are *equivalent relative to* $\mathbf{Y}$ if there is a finite sequence of monoid pictures $\mathbb{P} = \mathbb{P}_0, \mathbb{P}_1, \cdots, \mathbb{P}_m = \mathbb{P}'$ where, for $1 \leq i \leq m$, the monoid picture $\mathbb{P}_i$ is obtained from the picture $\mathbb{P}_{i-1}$ either by the insertion, deletion and replacement operations of the previous paragraph or else by inserting or deleting a subpicture of the form $W.\mathbb{Y}.V$ or of the form $W.\mathbb{Y}^{-1}.V$ where $W, V \in F(X)$ and $\mathbb{Y} \in \mathbf{Y}$. By definition, a set $\mathbf{Y}$ of spherical monoid pictures over $\mathcal{P}$ is a *trivializer of* $\mathcal{D}(\mathcal{P})$ if every spherical monoid picture is equivalent to an empty picture relative to $\mathbf{Y}$. By [13, Theorem 5.1], if $\mathbf{Y}$ is a trivializer for the Squier complex, then the elements of $\mathbf{Y}$ generate the first homology group of the Squier complex. The trivializer is also called a set of generating pictures. Some examples and more details of the trivializer can be found in [2], [3], [4], [9], [12], [13], [15] and [17].

For any monoid picture $\mathbb{P}$ over $\mathcal{P}$ and for any $R \in \mathbf{r}$, $\exp_R(\mathbb{P})$ denotes the *exponent sum* of R in $\mathbb{P}$ which is the number of positive discs labelled by R_+, minus the number of negative discs labelled by R_-. For a non-negative integer n, $\mathcal{P}$ is said to be *n-Cockcroft* if $\exp_R(\mathbb{P}) \equiv 0 \pmod n$, (where congruence (mod 0) is taken to be equality) for all $R \in \mathbf{r}$ and for all spherical pictures $\mathbb{P}$ over $\mathcal{P}$. Then a monoid $\mathcal{M}$ is said to be *n-Cockcroft* if it admits an n-Cockcroft presentation. In fact to verify the n-Cockcroft property, it is enough to check for pictures $\mathbb{P} \in \mathbf{Y}$, where $\mathbf{Y}$ is a trivializer (see [12], [13]). The 0-Cockcroft property is usually just called Cockcroft. In general we take n to be equal to 0 or a prime p. Examples of monoid presentations with Cockcroft and p-Cockcroft properties can be found in [3]. We note that the group case of these properties can be found in [10].

In group theory, the homological concept of efficiency has been widely studied. In [1], the authors defined efficiency for *finite* semigroups and hence for *finite* monoids. The following definition for not necessarily finite monoids follows [3] and is equivalent to the definition in [1] when the monoids are finite. For an abelian group G, $rk_{\mathbb{Z}}(G)$ denotes the $\mathbb{Z}$-rank of the torsion free part of G and $d(G)$ means the minimal number of generators of G. Suppose that $\mathcal{P} = [X; \mathbf{r}]$ is a finite presentation for a monoid $\mathcal{M}$. Then the *Euler characteristic*, $\chi(\mathcal{P})$ is defined by $\chi(\mathcal{P}) = 1 - |X| + |\mathbf{r}|$ and $\delta(\mathcal{M})$ is defined by $\delta(\mathcal{M}) = 1 - rk_{\mathbb{Z}}(H_1(\mathcal{M})) + d(H_2(\mathcal{M}))$. In unpublished work, S. Pride has shown that

$\chi(\mathcal{P}) \geq \delta(\mathcal{M})$. With this background, we define the finite monoid presentation $\mathcal{P}$ to be *efficient* if $\chi(\mathcal{P}) = \delta(\mathcal{M})$ and we define the monoid $\mathcal{M}$ to be *efficient* if it has an efficient presentation. Moreover a presentation $\mathcal{P}_0$ for $\mathcal{M}$ is called *minimal* if $\chi(\mathcal{P}_0) \leq \chi(\mathcal{P})$, for all presentations $\mathcal{P}$ of $\mathcal{M}$. There is also interest in finding *inefficient* finitely presented monoids since if we can find a minimal presentation $\mathcal{P}_0$ for a monoid $\mathcal{M}$ such that $\mathcal{P}_0$ is not efficient then we have $\chi(\mathcal{P}') \geq \chi(\mathcal{P}_0) > \delta(\mathcal{M})$, for all presentations $\mathcal{P}'$ defining the same monoid $\mathcal{M}$. Thus there is no efficient presentation for $\mathcal{M}$, that is, $\mathcal{M}$ is not an efficient monoid.

The following result is also an unpublished result by S. Pride. We will use this result rather than making more direct computations of homology for monoids. Kilgour and Pride prove the analogous result for groups in [10] and credit an earlier proof by Epstein, [6].

Theorem 1.1 *Let $\mathcal{P}$ be a monoid presentation. Then $\mathcal{P}$ is efficient if and only if it is p-Cockcroft for some prime p.*

The definition and a standard presentation for the semi-direct product of two monoids can be found in [3], [14], [16] or [17]. Let A and K be arbitrary monoids with associated presentations $\mathcal{P}_A = [X\,;\,\mathbf{r}]$ and $\mathcal{P}_K = [Y\,;\,\mathbf{s}]$, respectively. Let $M = K \rtimes_\theta A$ be the corresponding semi-direct product of these two monoids where θ is a monoid homomorphism from A to $End(K)$. (We note that the reader can find some examples of monoid endomorphisms in [5]). The elements of M can be regarded as ordered pairs (a, k) where $a \in A$, $k \in K$ with multiplication given by $(a, k)(a', k') = (aa', (k\theta_{a'})k')$. The monoids A and K are identified with the submonoids of M having elements $(a, 1)$ and $(1, k)$, respectively. We want to define standard presentations for M. For every $x \in X$ and $y \in Y$, choose a word, which we denote by $y\theta_x$, on Y such that $[y\theta_x] = [y]\theta_{[x]}$ as an element of K. To establish notation, let us denote the relation $yx = x(y\theta_x)$ on $X \cup Y$ by T_{yx} and write $\mathbf{t}$ for the set of relations T_{yx}. Then, for any choice of the words $y\theta_x$,

$$\mathcal{P}_M = [Y,\, X\ ;\ \mathbf{s},\, \mathbf{r},\, \mathbf{t}] \tag{1}$$

is a standard monoid presentation for the semi-direct product M.

In [17], Wang constructed a finite trivializer set for the standard presentation $\mathcal{P}_M$, as given in (1), for the semi-direct product M. We will essentially follow [3] in describing this trivializer set using spherical pictures and certain non-spherical subpictures of these.

If $W = y_1 y_2 \cdots y_m$ is a positive word on Y, then for any $x \in X$, we denote the word $(y_1\theta_x)(y_2\theta_x)\cdots(y_m\theta_x)$ by $W\theta_x$. If $U = x_1 x_2 \cdots x_n$ is a positive word on X, then for any $y \in Y$, we denote the word $(\cdots((y\theta_{x_1})\theta_{x_2})\theta_{x_3}\cdots)\theta_{x_n})$ by $y\theta_U$ and this can be represented by a monoid picture, say $\mathbb{A}_{U,y}$, as in Figure 1 (see also [3]). For $y \in Y$ and the relation $R_+ = R_-$ in the relation set $\mathbf{r}$, we have two important special cases, $\mathbb{A}_{R_+,y}$ and $\mathbb{A}_{R_-,y}$, of this consideration. We should note that these non-spherical pictures consist of only T_{yx}-discs $(x \in X)$.

Let $S \in \mathbf{s}$, $x \in X$. Since $[S_+\theta_x]_{\mathcal{P}_K} = [S_-\theta_x]_{\mathcal{P}_K}$, there is a non-spherical picture, say $\mathbb{B}_{S,x}$, over $\mathcal{P}_K$ with $\iota(\mathbb{B}_{S,x}) = S_+\theta_x$ and $\tau(\mathbb{B}_{S,x}) = S_-\theta_x$.

Let $R_+ = R_-$ be a relation $R \in \mathbf{r}$ and $y \in Y$. Since θ is a homomorphism, by our definition for $y\theta_U$, we have that $y\theta_{R_+}$ and $y\theta_{R_-}$ must represent the same element of the monoid K. That is, $[y\theta_{R_+}]_{\mathcal{P}_K} = [y\theta_{R_-}]_{\mathcal{P}_K}$. Hence there is a non-spherical picture over $\mathcal{P}_K$ which we denote by $\mathbb{C}_{y,\theta_R}$ with $\iota(\mathbb{C}_{y,\theta_R}) = y\theta_{R_+}$ and $\tau(\mathbb{C}_{y,\theta_R}) = y\theta_{R_-}$.

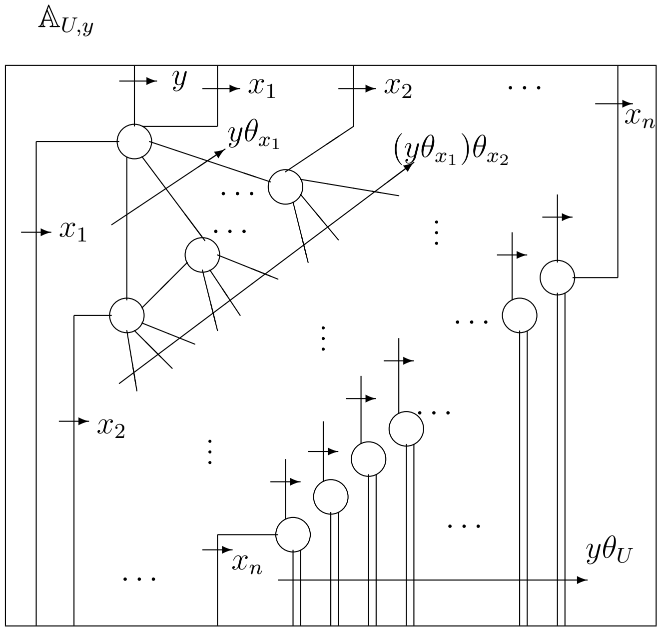

Figure 1:

In fact there may be many different ways to construct the pictures $\mathbb{B}_{S,x}$ and $\mathbb{C}_{y,\theta_R}$. These pictures must exist, but they are not unique. On the other hand the picture $\mathbb{A}_{U,y}$ will depend upon our choices for words $y\theta_x$, but this is unique once these choices are made.

After all, for $x \in X$, $y \in Y$, $R \in \mathbf{r}$ and $S \in \mathbf{s}$, one can construct spherical monoid pictures, say $\mathbb{P}_{S,x}$ and $\mathbb{P}_{R,y}$, by using the non-spherical pictures $\mathbb{B}_{S,x}$, $\mathbb{A}_{R_+,y}$, $\mathbb{A}_{R_-,y}$ and $\mathbb{C}_{y,\theta_R}$ (see [3, Figure 2] for the details).

Let $\mathbf{X_A}$ and $\mathbf{X_K}$ be trivializer sets of $\mathcal{D}(\mathcal{P}_A)$ and $\mathcal{D}(\mathcal{P}_K)$, respectively. Also, let

$$\mathbf{C_1} = \{\mathbb{P}_{S,x} \ : \ S \in \mathbf{s},\ x \in X\} \text{ and } \mathbf{C_2} = \{\mathbb{P}_{R,y} \ : \ R \in \mathbf{r},\ y \in Y\}.$$

The proof of the following lemma can be found in [17].

Lemma 1.2 *Suppose that $M = K \rtimes_\theta A$ is a semi-direct product with associated presentation $\mathcal{P}_M$, as in* (1). *Then a trivializer set of $\mathcal{D}(\mathcal{P}_M)$, say $\mathbf{X_M}$, is*

$$\mathbf{X_A} \cup \mathbf{X_K} \cup \mathbf{C_1} \cup \mathbf{C_2}.$$

Now, by using Lemma 1.2, we get the following main result in [3].

Theorem 1.3 [**3, Theorem 3.1**] *Let p be a prime or* 0. *Then the presentation $\mathcal{P}_M$, as in* (1), *is p-Cockcroft if and only if the following conditions hold.*

(*i*) *$\mathcal{P}_A$ and $\mathcal{P}_K$ are p-Cockcroft,*

(*ii*) $\exp_y(S) \equiv 0$ *(mod p) for all $S \in \mathbf{s}$, $y \in Y$,*

(*iii*) $\exp_S(\mathbb{B}_{S,x}) \equiv 1$ *(mod p) for all $S \in \mathbf{s}$, $x \in X$,*

(*iv*) $\exp_S(\mathbb{C}_{y,\theta_R}) \equiv 0$ *(mod p) for all $S \in \mathbf{s}$, $y \in Y$, $R \in \mathbf{r}$,*

(*v*) $\exp_{T_{yx}}(\mathbb{A}_{R_+,y}) \equiv \exp_{T_{yx}}(\mathbb{A}_{R_-,y})$ *(mod p) for all $R \in \mathbf{r}$, $y \in Y$ and $x \in X$.*

2 The p-Cockcroft property for the semi-direct products of finite cyclic monoids

In this section we will give necessary and sufficient conditions for the presentation of the semi-direct product of two finite cyclic monoids to be p-Cockcroft (p a prime). We first note that most of the following materials given in this section can also be found in [3]. We also note that some of the fundamental materials about the finite cyclic monoids can be found in [8] (see "Monogenic semigroups").

Let A and K be two finite cyclic monoids with the presentations (by [3], [8])

$$\mathcal{P}_A = [x\,;\, x^{\mu} = x^{\lambda}] \quad \text{and} \quad \mathcal{P}_K = [y\,;\, y^k = y^l] \tag{2}$$

respectively, where $l, k, \lambda, \mu \in \mathbb{Z}^+$ such that $l < k$ and $\lambda < \mu$.

We can give the following lemma for a trivializer set of the finite cyclic monoids (see [3]).

Lemma 2.1 *Let K be the finite cyclic monoid with a presentation $\mathcal{P}_K$. Then a trivializer set $\mathbf{X_K}$ of the Squier complex $\mathcal{D}(\mathcal{P}_K)$ is given by the pictures $\mathbb{P}^m_{k,l}$ $(1 \leq m \leq k-1)$, as in Figure 2.*

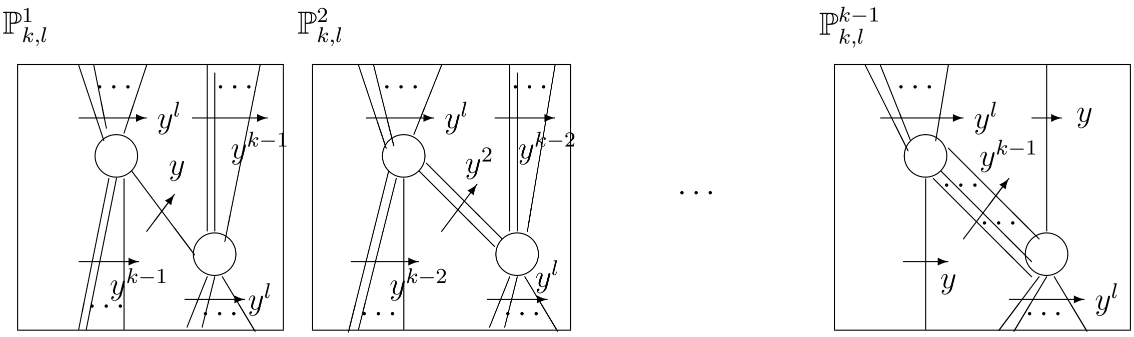

Figure 2:

Let ψ_i $(0 \leq i \leq k-1)$ be an endomorphism of K. Then we have a mapping

$$x \longrightarrow End(K), \quad x \longmapsto \psi_i.$$

In fact this induces a homomorphism $\theta : A \longrightarrow End(K)$, $x \longmapsto \psi_i$ if and only if $\psi_i^{\mu} = \psi_i^{\lambda}$. Since ψ_i^{μ} and ψ_i^{λ} are equal if and only if they agree on the generator y of K, we must have

$$[y^{i^{\mu}}] = [y^{i^{\lambda}}]. \tag{3}$$

We then have the semi-direct product $M = K \rtimes_{\theta} A$ and, by [3], have a standard presentation

$$\mathcal{P}_M = [y,\, x\,;\, S,\, R,\, T_{yx}], \tag{4}$$

as in (1), for the monoid M where

$$S : y^k = y^l, \;\; R : x^{\mu} = x^{\lambda} \;\text{ and }\; T_{yx} : yx = xy^i.$$

At the rest of the paper we will assume that the equality (3) holds when we talk about the semi-direct product M of K by A.

The subpicture $\mathbb{B}_{S,x}$ can be drawn as in Figure 3-(a) and in fact, by considering this subpicture, we have the following lemma .

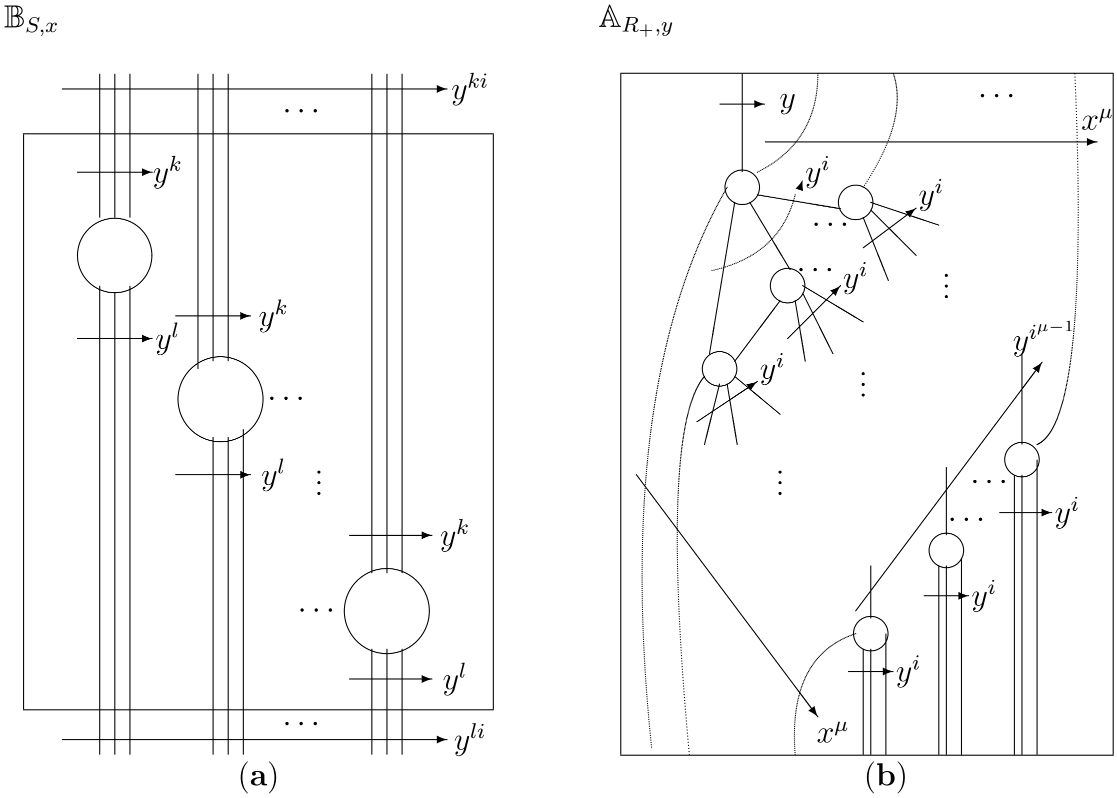

Figure 3:

Lemma 2.2 $\exp_S(\mathbb{B}_{S,x}) = i$.

As it seen in Figure 3-(b), we also have the subpicture $\mathbb{A}_{R_+,y}$ (and similarly $\mathbb{A}_{R_-,y}$) with

$$\exp_{T_{yx}}(\mathbb{A}_{R_+,y}) = 1 + i + i^2 + \cdots + i^{\mu-1} = \frac{i^\mu - 1}{i - 1}$$

and

$$\exp_{T_{yx}}(\mathbb{A}_{R_-,y}) = 1 + i + i^2 + \cdots + i^{\lambda-1} = \frac{i^\lambda - 1}{i - 1}.$$

By equality (3), we must have $[y^{i^\mu}] = [y^{i^\lambda}]$. Hence, by [3], the subpicture $\mathbb{C}_{y,\theta_R}$ with

$$\iota(\mathbb{C}_{y,\theta_R}) = y^{i^\mu}, \quad \tau(\mathbb{C}_{y,\theta_R}) = y^{i^\lambda} \quad \text{and} \quad \exp_S(\mathbb{C}_{y,\theta_R}) = \frac{i^\mu - i^\lambda}{k - l}$$

can be depicted as in Figure 4.

Hence the generating pictures $\mathbb{P}_{S,x}$ and $\mathbb{P}_{R,y}$ can be depicted as in Figure 5.

We will use the following special case of the main result in [3]. In fact this following theorem is a consequence of Theorem 1.3. Let $d = k - l$, $n = i - 1$, $t = i^\mu - i^\lambda$.

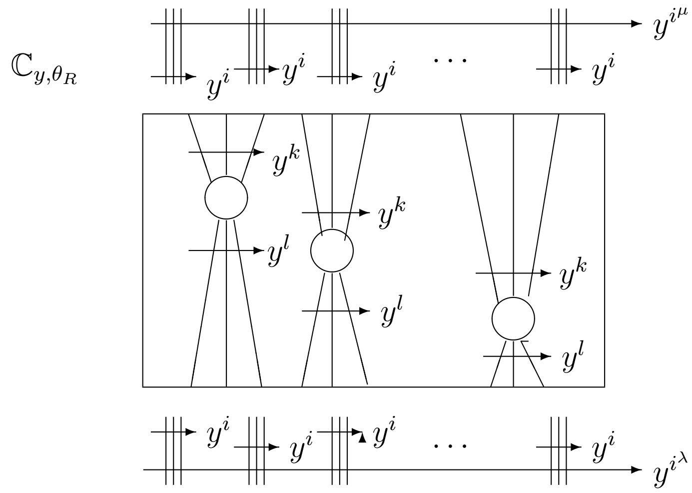

Figure 4:

Theorem 2.3 *Let p be a prime. Suppose that $K \rtimes_\theta A$ is a monoid with the associated monoid presentation $\mathcal{P}_M$, as in* (4)*. Then $\mathcal{P}_M$ is p-Cockcroft if and only if*

$$p \mid d,\ p \mid n,\ p \mid \frac{t}{d},\ p \mid \frac{t}{n}.$$

Proof. To prove the this result we will check the conditions of Theorem 1.3 hold.

(i) By Lemma 2.1, trivializer sets $\mathbf{X_A}$ and $\mathbf{X_K}$ of the Squier complexes $\mathcal{D}(\mathcal{P}_A)$ and $\mathcal{D}(\mathcal{P}_K)$ respectively, can be given as in Figure 2. Thus it can be seen that $\mathcal{P}_A$ and $\mathcal{P}_K$ are p-Cockcroft (in fact Cockcroft), and then the condition (i) holds.

(ii) $\exp_y(S) = k - l$ so, for (ii) to hold, we must have $p \mid k - l$.

(iii) To make (iii) hold, we need $i \equiv 1 \pmod p$, so that $p \mid i - 1$.

(iv) For the subpicture $\mathbb{C}_{y,\theta_R}$, we must have $p \mid \frac{i^\mu - i^\lambda}{k-l}$, to make ($iv$) hold.

(v) Also, to make (v) hold, we need $\frac{i^\mu - 1}{i-1} \equiv \frac{i^\lambda - 1}{i-1} \pmod p$, or equivalently, $\frac{i^\mu - i^\lambda}{i-1} \equiv 0 \pmod p$ since

$$exp_{T_{yx}}(\mathbb{A}_{R,y}) = exp_{T_{yx}}(\mathbb{A}_{R_+,y}) - exp_{T_{yx}}(\mathbb{A}_{R_-,y}).$$

Conversely suppose that the conditions of the theorem hold. Then, by the meaning of the trivializer set $\mathbf{X_M}$, it is easy to see that $\mathcal{P}_M$ is p-Cockcroft where p is a prime.

Hence the result. $\diamondsuit$

The examples and consequences of Theorem 2.3 can be found in [3].

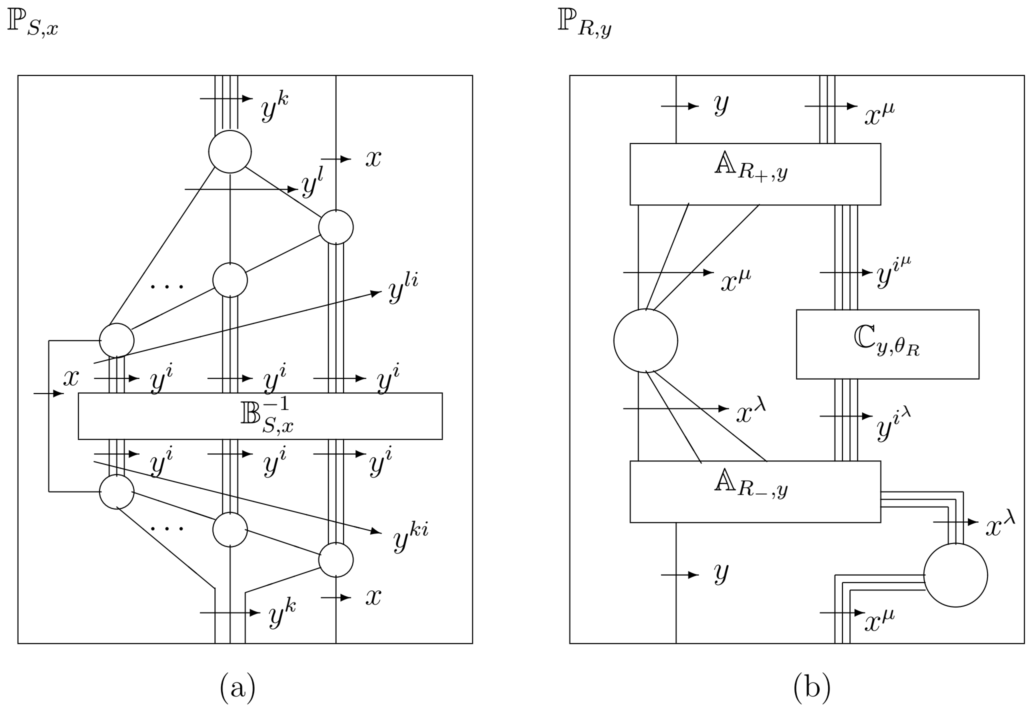

Figure 5:

3 The main theorem

By Theorems 1.3 and 2.3, one can say that the monoid presentation $\mathcal{P}_M$, as in (4), is efficient if and only if there is a prime p such that

$$exp_y(S) \equiv 0 \ (\text{mod } p), \qquad exp_S(\mathbb{B}_{S,x}) \equiv 1 \ (\text{mod } p),$$
$$exp_{T_{yx}}(\mathbb{C}_{y,\theta_R}) \equiv 0 \ (\text{mod } p), \qquad exp_{T_{yx}}(\mathbb{A}_{R,y}) \equiv 0 \ (\text{mod } p).$$

In particular, if we choose $exp_S(\mathbb{B}_{S,x}) = 0$ or 2 (or, by Lemma 2.2, $i = 0$ or $i = 2$) then $\mathcal{P}_M$ will be inefficient.

Let $d = exp_y(S) = k - l$. We note that, by the meaning of finite cyclic monoids, the value of d cannot be equal to 0.

The main result of this paper is the following.

Theorem 3.1 *Let M be the semi-direct product of K by A, and let*

$$\mathcal{P}_M = [y,\, x \ ; \ y^k = y^l,\, x^\mu = x^\lambda,\, yx = xy^i]$$

be a standard presentation of M where $l, k, \lambda, \mu, i \in \mathbb{Z}^+$ and $l < k$, $\lambda < \mu$. If $i = 2$ and d is not even and not equal to 1, then $\mathcal{P}_M$ is **minimal** *but* **inefficient**.

4 The preliminaries

We should note that some of the similar preliminary material can also be found in [4].

Let M be a monoid with a presentation $\mathcal{P} = [Y \ ; \ \mathbf{s}]$, and let $P^{(l)} = \bigoplus_{S \in \mathbf{s}} \mathbb{Z}Me_S$ be the free left $\mathbb{Z}M$-module with bases $\{e_S : S \in \mathbf{s}\}$. For an atomic monoid picture, say

$\mathbb{A} = (U, S, \varepsilon, V)$ where $U, V \in F(\mathbf{y})$, $S \in \mathbf{s}$, $\varepsilon = \pm 1$, the left evaluation of the positive atomic monoid picture $\mathbb{A}$ is defined by $eval^{(l)}(\mathbb{A}) = \varepsilon \overline{U} e_S \in P^{(l)}$ where $\overline{U} \in M$. For any spherical monoid picture $\mathbb{P} = \mathbb{A}_1\mathbb{A}_2 \cdots \mathbb{A}_n$, where each $\mathbb{A}_i$ is an atomic picture for $i = 1, 2, \cdots, n$, we then define

$$eval^{(l)}(\mathbb{P}) \;\; = \;\; \sum_{i=1}^{n} eval^{(l)}(\mathbb{A}_i) \in P^{(l)}.$$

We let $\delta_{\mathbb{P},S}$ be the coefficient of e_S in $eval^{(l)}(\mathbb{P})$, so we can write

$$eval^{(l)}(\mathbb{P}) = \sum_{S \in \mathbf{s}} \delta_{\mathbb{P},S} e_S \in P^{(l)}.$$

Let $I_2^{(l)}(\mathcal{P})$ be the 2-sided ideal of $\mathbb{Z}M$ generated by the set

$$\{\delta_{\mathbb{P},S} : \mathbb{P} \text{ is a spherical monoid picture}, S \in \mathbf{s}\}.$$

Then this ideal is called the *second Fox ideal* of $\mathcal{P}$.

The fact of the following lemma has been also discussed in [4].

Lemma 4.1 *If $\mathbf{Y}$ is a trivializer of $\mathcal{D}(\mathcal{P})$ then second Fox ideal is generated by the set $\{\delta_{\mathbb{P},S} : \mathbb{P} \in \mathbf{Y}, S \in \mathbf{s}\}$.*

The concept of the second Fox ideals will be needed for the following theorem which can be thought as a *test of minimality of monoid presentations* and has been proved in an unpublished work by S. Pride (as depicted also in [4]). We remark that the case of the group presentation version of this result has been firstly proved by M. Lustig in [11].

Theorem 4.2 *Let $\mathbf{Y}$ be a trivializer of $\mathcal{D}(\mathcal{P})$ and let ψ be a ring homomorphism from $\mathbb{Z}M$ into the ring of all $k \times k$ matrices over a commutative ring L with 1, for some $k \geq 1$, and suppose $\psi(1) = I_{k\times k}$. If $\psi(\delta_{\mathbb{P},S}) = 0$ for all $\mathbb{P} \in \mathbf{Y}$, $S \in \mathbf{s}$ then $\mathcal{P}$ is minimal.*

Theorem 4.2 can be restated by *if there is a ring homomorphism ψ as above such that $I_2^{(l)}(\mathcal{P})$ is contained in the kernel of ψ, then $\mathcal{P}$ is minimal.*

Let us suppose that both K and A be the finite cyclic monoids with presentations $\mathcal{P}_K$ and $\mathcal{P}_A$ as in (2), and let M be the semi-direct product of K by A with a presentation as in (4).

Let us consider the picture $\mathbb{P}_{S,x}$, as in Figure 5-(a).

For the generator y, let us assume that $\dfrac{\partial}{\partial y}$ denotes the Fox derivation with respect to y, and let $\dfrac{\partial^M}{\partial y}$ be the composition

$$\mathbb{Z}F(y) \xrightarrow{\frac{\partial}{\partial y}} \mathbb{Z}F(y) \longrightarrow \mathbb{Z}M,$$

where $F(y)$ is the free monoid on y. Also, for the relator $S : y^k = y^l$, let us define $\dfrac{\partial^M S}{\partial y}$ to be

$$\frac{\partial^M S_+}{\partial y} - \frac{\partial^M S_-}{\partial y}.$$

In fact, for this y, the left evaluations of the positive atomic monoid pictures in $\mathbb{P}_{S,x}$ containing a T_{yx} disc are

$$\overline{1}e_{T_{yx}},\ \overline{y}e_{T_{yx}},\ \overline{y}^2e_{T_{yx}},\ \cdots,\overline{y}^{l-1}e_{T_{yx}},$$

and the left evaluations of the negative atomic monoid pictures in $\mathbb{P}_{S,x}$ containing a T_{yx} disc are

$$-\overline{1}e_{T_{yx}},\ -\overline{y}e_{T_{yx}},\ -\overline{y}^2e_{T_{yx}},\ \cdots,-\overline{y}^{k-1}e_{T_{yx}}.$$

Thus the coefficient of $e_{T_{yx}}$ in $eval^{(l)}(\mathbb{P}_{S,x})$ is

$$\overline{1}+\overline{y}+\overline{y}^2+\cdots+\overline{y}^{l-1}-(\overline{1}+\overline{y}+\overline{y}^2+\cdots+\overline{y}^{k-1})=\frac{\partial^M S}{\partial y}. \tag{5}$$

Proposition 4.3 *The second Fox ideal $I_2^{(l)}(\mathcal{P}_M)$ of $\mathcal{P}_M$ is generated by the elements*

$$\begin{array}{ll} 1-\bar{x}(eval^{(l)}(\mathbb{B}_{S,x})), & \dfrac{\partial^M S}{\partial y}, \\ eval^{(l)}(\mathbb{A}_{R_+,x})-eval^{(l)}(\mathbb{A}_{R_-,x}), & eval^{(l)}(\mathbb{C}_{y,\theta_R}), \\ 1-\overline{y}^{k-1},\ 1-\overline{y}^{k-2},\cdots,1-\overline{y} \quad \text{and} & 1-\overline{x}^{\mu-1},\ 1-\overline{x}^{\mu-2},\ \cdots,1-\overline{x}. \end{array}$$

Proof. In the proof, for simplicity, we shall not use the characters with “bar” in the evaluations.

By Lemma 1.2, since $\mathcal{D}(\mathcal{P}_M)$ has a trivializer $\mathbf{X_M}$ consisting of the sets $\mathbf{X_A}$, $\mathbf{X_K}$, $\mathbf{C_1}$ and $\mathbf{C_2}$ where $\mathbf{X_A}$, $\mathbf{X_K}$ are the trivializer sets of $\mathcal{D}(\mathcal{P}_A)$ and $\mathcal{D}(\mathcal{P}_K)$ respectively and $\mathbf{C_1}$, $\mathbf{C_2}$ consist of the single pictures $\mathbb{P}_{S,x}$, $\mathbb{P}_{R,y}$ respectively (see Figures 2 and 5), we need to calculate $eval^{(l)}(\mathbb{P}_{S,x})$, $eval^{(l)}(\mathbb{P}_{R,y})$, $eval^{(l)}(\mathbb{P}^m_{k,l})$ $(1\le m\le k-1)$ and $eval^{(l)}(\mathbb{P}^n_{\mu,\lambda})$ $(1\le n\le \mu-1)$. So we have

$$\begin{aligned} eval^{(l)}(\mathbb{P}_{S,x}) &= \delta_{\mathbb{P}_{S,x},S}e_S+\delta_{\mathbb{P}_{S,x},T_{yx}}e_{T_{yx}} \\ &= (1-x(eval^{(l)}(\mathbb{B}_{S,x})))e_S+(\frac{\partial^M S}{\partial y})e_{T_{yx}} \text{ by (5)} \\ eval^{(l)}(\mathbb{P}_{R,y}) &= \delta_{\mathbb{P}_{R,y},T_{yx}}e_{T_{yx}}+\delta_{\mathbb{P}_{R,y},R}e_R+\delta_{\mathbb{P}_{R,y},S}e_S \\ &= (eval^{(l)}(\mathbb{A}_{R_+,y})-eval^{(l)}(\mathbb{A}_{R_-,y}))e_{T_{yx}}+(1-y)e_R+(eval^{(l)}(\mathbb{C}_{y,\theta_R}))e_S. \end{aligned}$$

Also, for each $1\le m\le k-1$ and $1\le n\le \mu-1$,

$$eval^{(l)}(\mathbb{P}^m_{k,l})=\delta_{\mathbb{P}^m_{k,l},S}e_S \quad \text{and} \quad eval^{(l)}(\mathbb{P}^n_{\mu,\lambda})=\delta_{\mathbb{P}^n_{\mu,\lambda},R}e_R,$$

where $\delta_{\mathbb{P}^m_{k,l},S}=1-y^{k-m}$ and $\delta_{\mathbb{P}^n_{\mu,\lambda},R}=1-x^{\mu-n}$.

Thus, by Lemma 4.1, we get the result as required. $\diamondsuit$

Let

$$aug:\mathbb{Z}M\longrightarrow\mathbb{Z},\quad b\longmapsto 1$$

be the augmentation map. We then have the following lemma.

Lemma 4.4 *We have the following equalities.*

1) $aug(eval^{(l)}(\mathbb{B}_{S,x})) = \exp_S(\mathbb{B}_{S,x})$.

2) $aug(\frac{\partial^M S}{\partial y}) = \exp_y(S) = k - l$.

3) $aug(eval^{(l)}(\mathbb{A}_{R_+,y}) - eval^{(l)}(\mathbb{A}_{R_-,y})) = \exp_{T_{yx}}(\mathbb{P}_{R,y}) = \frac{i^\mu - i^\lambda}{i-1}$.

4) $aug(eval^{(l)}(\mathbb{C}_{y,\theta_R})) = \exp_S(\mathbb{P}_{R,y}) = \frac{i^\mu - i^\lambda}{k-l}$.

5) $aug(eval^{(l)}(\mathbb{P}^m_{k,l})) = 0$ *and* $aug(eval^{(l)}(\mathbb{P}^n_{\mu,\lambda})) = 0$, *for each* $1 \leq m \leq k-1$ *and* $1 \leq n \leq \mu - 1$.

Proof. The proofs of the equalities given in 1) and 2) can be found in [4].

Proof of 3):

We can write

$$\begin{aligned} eval^{(l)}(\mathbb{A}_{R_+,y}) &= \varepsilon_1 \overline{W_1} e_{T_{yx}} + \varepsilon_2 \overline{W_2} e_{T_{yx}} + \cdots + \varepsilon_j \overline{W_j} e_{T_{yx}}, \\ eval^{(l)}(\mathbb{A}_{R_-,y}) &= \gamma_1 \overline{U_1} e_{T_{yx}} + \gamma_2 \overline{U_2} e_{T_{yx}} + \cdots + \gamma_q \overline{U_q} e_{T_{yx}} \end{aligned}$$

where, for $1 \leq a \leq j$, $1 \leq b \leq q$, each $\varepsilon_a = 1$, $\gamma_b = -1$ and each of W_a, U_b is a certain word on the set $\{x, y\}$. In the right hand side of the above equalities, each term $\varepsilon_a \overline{W_a} e_{T_{yx}}$ and $\gamma_q \overline{U_q} e_{T_{yx}}$ corresponds to a single T_{yx}-disc and, in fact, the value of each ε_a and γ_b gives the sign of this single T_{yx}-disc. Therefore the sum of the ε_a's and γ_b's, that is, $aug(eval^{(l)}(\mathbb{A}_{R_+,y}) - eval^{(l)}(\mathbb{A}_{R_-,y}))$ must give the exponent sum of the T_{yx}-discs in the picture $\mathbb{P}_{R,y}$, as required since the T_{yx}-discs can only be occured in the subpictures $\mathbb{A}_{R_+,y}$ and $\mathbb{A}_{R_-,y}$.

Proof of 4):

We have just the S-discs in the subpicture $\mathbb{C}_{y,\theta_R}$ (see Figure 4) in $\mathbb{P}_{R,y}$. Then, by writing

$$eval^{(l)}(\mathbb{C}_{y,\theta_R}) = \overline{x}^\mu (\epsilon_1 \overline{V_1} e_S + \epsilon_2 \overline{V_2} e_S + \cdots + \epsilon_g \overline{V_g} e_S)$$

and adapting the proof of 3) into this case, we get the result.

Proof of 5):

For each $1 \leq m \leq k-1$ and $1 \leq n \leq \mu - 1$, since each of $\mathbb{P}^m_{k,l}$ and $\mathbb{P}^n_{\mu,\lambda}$ contains just two S-discs and R-discs (which one is positive and the other is negative) respectively, we write

$$eval^{(l)}(\mathbb{P}^m_{k,l}) = -\overline{W^m_1} e_S + \overline{W^m_2} e_S \quad \text{and} \quad eval^{(l)}(\mathbb{P}^n_{\mu,\lambda}) = -\overline{U^n_1} e_R + \overline{U^n_2} e_R,$$

where W^m_i's are words on y and U^n_j's are words on x ($1 \leq i, j \leq 2$). Again as in the previous cases, by considering the each term in the above equalities, we get the sign of this single S-disc and single R-disc. Then the sum of the these signs, in other words, augmentation of the evaluation of each picture must give the exponent sum of S and R-discs. That is,

$$aug(eval^{(l)}(\mathbb{P}^m_{k,l})) = \exp_S(\mathbb{P}^m_{k,l}) = -1 + 1 = 0 = \exp_S(\mathbb{P}^n_{\mu,\lambda}) = aug(eval^{(l)}(\mathbb{P}^n_{\mu,\lambda})),$$

as required.

Hence the result. $\diamondsuit$

5 Proof of the main Theorem

Suppose that $d = k - l$ is not equal to 1 and $2n$ ($n \in \mathbb{Z}^+$). Let $\mathbb{Z}_d$ defines $\mathbb{Z}$ (mod d) while $d \neq 0$. (Recall that d cannot be equal to 0). Suppose also that $\exp_S(\mathbb{B}_{S,x}) = 2$ (or, equivalently, $i = 2$ by Lemma 2.2).

Let $M_{\mu,\lambda}$ defines the finite cyclic monoid generated by x. Let us consider the homomorphism from M onto $M_{\mu,\lambda}$ defined by

$$y \longmapsto 1, \quad x \longmapsto x.$$

This induces a ring homomorphism

$$\gamma : \mathbb{Z}M \longrightarrow M_{\mu,\lambda}[x].$$

We note that the restriction of γ to the subring $\mathbb{Z}K$ of $\mathbb{Z}M$ is just the augmentation map $aug : \mathbb{Z}K \longrightarrow \mathbb{Z}$. Thus, by Lemma 4.4, the image of $I_2^{(l)}(\mathcal{P}_M)$ under γ is the ideal of $M_{\mu,\lambda}[x]$ generated by

$$1 - \bar{x}(\exp_S(\mathbb{B}_{S,x})) = 1 - 2\bar{x}, \ \exp_y(S), \ \exp_{T_{yx}}(\mathbb{P}_{R,y}) \text{ and } \exp_S(\mathbb{P}_{R,y}).$$

Let η be the composition of γ and the mapping

$$M_{\mu,\lambda}[x] \longrightarrow \mathbb{Z}_d[x], \quad x \longmapsto x, \, n \longmapsto \overline{n} \ (n \in \mathbb{Z}),$$

where $\overline{n}$ is n (mod d). Then, since $\exp_y(S) = d \equiv 0$ (mod d), $\exp_{T_{yx}}(\mathbb{P}_{R,y}) \equiv 0$ (mod d), $\exp_S(\mathbb{P}_{R,y}) \equiv 0$ (mod d), we get

$$\begin{aligned} \eta(I_2^{(l)}(\mathcal{P}_M)) &= \ < 1 - \overline{2}\bar{x} > \\ &= I, \text{ say.} \end{aligned}$$

A quite similar proof for the following lemma can be found in [4].

Lemma 5.1

$$I \neq \mathbb{Z}_d[x].$$

Remark 5.2 *In the proof of Lemma* 5.1 *one can see that if* $d = 1$ *then* $I = \mathbb{Z}_d[x]$ *(see* [4, Lemma 4.6]*).*

Let ψ be the composition

$$\mathbb{Z}M \xrightarrow{\eta} \mathbb{Z}_d[x] \xrightarrow{\phi} \mathbb{Z}_d[x]/I,$$

where ϕ is the natural epimorphism. Then ψ sends $I_2^{(l)}(\mathcal{P}_M)$ to 0, and $\psi(1) = 1$. In other words, the images of the generators of $I_2(\mathcal{P}_M)$ are all 0 under ψ. That is,

$$\begin{aligned} \psi(1 - \bar{x}(eval^{(l)}(\mathbb{B}_{S,x}))) &= \phi\eta(1 - \bar{x}(eval^{(l)}(\mathbb{B}_{S,x}))) \\ &= \phi(1 - \bar{x}(\overline{\exp_S(\mathbb{B}_{S,x})}) \ \text{ since } \eta \text{ is a ring} \\ &\quad\ \text{homomorphism and by Lemma } 4.4 - 1) \\ &= \phi(1 - \bar{x}\overline{2}) \ \text{ since } \exp_S(\mathbb{B}_{S,x}) = 2 \\ &= 0, \end{aligned}$$

$$\begin{aligned}
\psi(\frac{\partial^M S}{\partial y}) &= \phi\eta(\frac{\partial^M S}{\partial y}) \\
&= \phi(\overline{\exp_y(S)}) \text{ since } \eta \text{ is a ring} \\
&\quad \text{homomorphism and by Lemma } 4.4-2) \\
&= \phi(0) \quad \text{since } \exp_y(S) = d \equiv 0 \pmod d \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\psi(eval^{(l)}(\mathbb{A}_{R_+,y}) - eval^{(l)}(\mathbb{A}_{R_-,y})) &= \phi\eta(eval^{(l)}(\mathbb{A}_{R_+,y}) - eval^{(l)}(\mathbb{A}_{R_-,y})) \\
&= \phi(\overline{\exp_{T_{yx}}(\mathbb{P}_{R,y})}) \quad \text{since } \eta \text{ is a ring} \\
&\quad \text{homomorphism and by Lemma } 4.4-3) \\
&= \phi(\overline{\frac{i^\mu - i^\lambda}{i-1}}) \\
&= \phi(0) \text{ since the equation (3) implies} \\
&\quad i^\mu - i^\lambda \equiv 0 \pmod p, \text{ and so } \frac{i^\mu - i^\lambda}{i-1} \equiv 0 \pmod d \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\phi(eval^{(l)}(\mathbb{C}_{y,\theta_R})) &= \phi\eta(eval^{(l)}(\mathbb{C}_{y,\theta_R})) \\
&= \phi(\overline{\exp_S(\mathbb{P}_{R,y})}) \text{ since } \eta \text{ is a ring} \\
&\quad \text{homomorphism and by Lemma } 4.4-4) \\
&= \phi(\overline{\frac{i^\mu - i^\lambda}{k-l}}) \\
&= \phi(0) \text{ by the equation (3)} \\
&= 0.
\end{aligned}$$

So, by Theorem 4.2 (Pride), $\mathcal{P}_M$ is minimal and so, by the definition, M is a minimal but inefficient monoid.

These above processes complete the proof of Theorem 3.1. $\diamondsuit$

Lemma 5.3 *Suppose that* $d = 2n$ $(n \in \mathbb{Z}^+)$. *Then* $I = \mathbb{Z}_d[x]$.

Proof. For simplicity, let us replace $\bar{x}$ by x and $\overline{2}$ by 2.

In the proof it is enough to show $2 \in I =< 1 - 2x >$. Because we certainly have $1 - 2x \in I$ and if $2 \in I$ then we must have $1 \in I$.

Thus let us take $1 - 2x \in I$. Then, by the meaning of $< 1 - 2x >$, we can write

$$\begin{aligned}
&2(n-1)(1-2x) \in I \Rightarrow 2(n-1) - 4(n-1)x \in I \Rightarrow \\
&2(n-1) - 4nx + 4x \in I \Rightarrow 2(n-1) \in I \\
&\text{since } 4nx = 0 \text{ and } 4x = 0 \text{ in } \mathbb{Z}_d[x].
\end{aligned}$$

Then,

$$
\begin{aligned}
& 2(n-2)(1-2x) \in I \Rightarrow 2(n-2) - 4(n-2)x \in I \Rightarrow \\
& 2(n-2) - 4(n-1-1)x \in I \Rightarrow 2(n-2) - (4(n-1)x - 4x) \in I \Rightarrow \\
& 2(n-2) - 2(2(n-1)x - 2x) \in I \Rightarrow 2(n-2) \in I \\
& \text{since, by the above calculation, } 2(n-1) \in I \Rightarrow 2(n-1)x \in I \\
& \text{and } 2x = 0 \text{ in } \mathbb{Z}_d[x] \\
\Rightarrow \quad & \cdots \text{ by iterating this procedure, we get } \cdots \Rightarrow \\
& 2 \in I \Rightarrow 1 \in I,
\end{aligned}
$$

as required. $\diamondsuit$

Remark 5.4 *Suppose that $i = 0$. Then the presentation $\mathcal{P}_M$, as in (4), can be writen*

$$\mathcal{P}_M = [y,\, x\ ;\ y^k = y^l,\, x^\mu = x^\lambda,\, yx = x]. \tag{6}$$

Then it is easy to see that there will not be any $\mathbb{B}_{S,x}$ and $\mathbb{C}_{y,\theta_R}$ subpictures. Also there is no need any restriction on $d = k - l$ since $i = 0$ and so the equality (3) always hold. However, by using same progress as in the proof of Theorem 3.1, we have

$$\eta(I_2^{(l)}(\mathcal{P}_M)) =< 1 >= I.$$

That means the minimality test (Theorem 4.2) used in this paper cannot work for this case since $1 \in I$ and so $I = \mathbb{Z}_d[x]$. Therefore it can be remained as a conjecture whether the presentation $\mathcal{P}_M$ given in (6) is minimal.

6 Some examples

In this section we will give two applications of Theorem 3.1.

Example 6.1 *Let us take $i = 2$, $k = 4t$, $l = t$, $\mu = 3t$ and $\lambda = t$ where t is an odd positive integer. Clearly the equality in (3) holds and so we obtain a presentation*

$$\mathcal{P}_M = [y,\, x\ ;\ y^{4t} = y^t,\, x^{3t} = x^t,\, yx = xy^2], \tag{7}$$

as in (4), for the monoid $M = K \rtimes_\theta A$. By Theorem 2.3, $\mathcal{P}_M$ is an inefficient presentation. Moreover $d = 3t \neq 2n$ ($n \in \mathbb{Z}^+$).

Thus as a consequence of Theorem 3.1, we have

Corollary 6.2 *For every positive odd integer t, the presentation $\mathcal{P}_M$ given in (7) is minimal but inefficient.*

Example 6.3 *Let $i = 2$, $k = 2t+1$, $l = 2s$ ($s < t$, $t, s \in \mathbb{Z}^+$), $\mu = k-l$, $\lambda = 1$. Therefore equality (3) satisfies and then we have a semi-direct product M with the presentation*

$$\mathcal{P}_M = [y,\, x\ ;\ y^{2t+1} = y^{2s},\, x^{k-l} = x,\, yx = xy^2]. \tag{8}$$

It is clear that $d = 2(t-s)+1 \neq 2n$ ($n \in \mathbb{Z}^+$). Also, by Theorem 2.3, $\mathcal{P}_M$ is an inefficient presentation.

As an application of Theorem 3.1, we have

Corollary 6.4 *For all $t, s \in \mathbb{Z}^+$ such that $s < t$, the presentation $\mathcal{P}_M$, as in (8), is minimal but inefficient.*

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